Invariant Equilibria and Classes of Equivalent Games

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Abstract

We consider classes of games for a fixed set of players with fixed strategy sets. For such classes, we explicate the concept of invariance, which is satisfied when the set of Nash equilibria and corresponding equilibrium payoffs are identical for each payoff function within the class. We introduce the condition superior payoff matching, which requires that at any given strategy profile, each player can match her highest payoff near that strategy profile across all games within that class. If a specific game satisfies superior payoff matching, then its equilibria are invariant within a class of games with smaller sets of non-maximal discontinuities. This condition can be used to prove existence of Nash equilibrium. Using invariance, this existence result applies to games that are not quasiconcave or better reply secure. Additionally, superior payoff matching is adapted to show results for invariant symmetric equilibrium and invariant mixed strategy equilibrium.

Key words: Invariance, Discontinuous games, Nash equilibrium, Sharing rules, Existence.

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1 Introduction

The standard approach to studying Nash equilibria is to begin with a game and analyze how the properties of that game affect its equilibria. In this paper, we offer an alternative approach: to consider a class of games and analyze how properties of the whole class affect the equilibria within that class. To that effect, we consider the concept of invariant Nash equilibria, which are the Nash equilibria that are shared by all games within a given class.[1] By focusing on the properties that differ across games within a class, we are able to derive some general characteristics that must be satisfied by invariant equilibria. This approach is particularly useful for understanding the influence of the specification of payoffs at points of discontinuity as well as for verifying existence of equilibrium in discontinuous games.

To provide some context to the contribution, our results can be immediately applied to extend results of many of the literatures on applications of discontinuous games, including the following recent contributions: *All-pay contests*: Siegel (2009, 2011); *Bertrand-Edgeworth price competition*: Allison and Lepore (2016); *Price competition with limited comparability:* Piccione and Spiegler (2012); *Catalog competition*: Page and Monteiro (2003), Monteiro and Page (2007, 2008); *Network attack and defense*: Kovenock and Roberson (2017); *Peer Pressure*: Calvó-Armengol and Jackson (2010); *Pre-marital investment*: Peters (2007) ; *Labor market search*: Galenianos and Kircher (2012); *Competitive matching*: Damiano and Li (2008).

Before we discuss the conditions and results, it is necessary to give some details of the environment we consider. As per usual, a game is a set of players, a strategy space for each player, and a payoff function specifying the outcome for each player given any combination of strategies. A class of games is a set of players, a strategy space for each player, and a set of possible payoff functions. Thus, for two games to be considered as members of the same class, they must have the same players and same strategy sets. Two games within a class may thus have wildly different payoffs and describe completely different settings. Each payoff function can be viewed as a composition between a mapping from actions to outcomes and a fixed preference over outcomes. From this perspective, the players each have a single preference over outcomes, and different payoffs correspond to different mappings from actions to outcomes. Many applications naturally yield a class of games that would warrant analysis. This multiplicity of payoff specifications may arise from modelling choices, but most prominently arises from the presence of discontinuities. Indeed, any game with discontinuous payoffs typically admits a continuum of possible payoffs at every point of discontinuity. In these cases, we define a game's associated class to be the set of payoff functions that includes all measurable combinations of possible payoffs. More specifically, a game's *associated class* consists of all measurable functions bounded between the upper and lower semicontinuous envelope of each player's payoff.

[1] This notion of invariance is borrowed from Jackson and Swinkles (2005), which verifies this invariance for the Nash equilibria of single and double private value auctions.

To be more precise, there are two notions of invariance that we are interested in: invariance of an equilibrium (or equilibria), and invariance of games or a class of games. An equilibrium strategy profile is invariant within a class of games if it is an equilibrium of all games within a class. A class of games is invariant if all games within the class have the same Nash equilibria.[2] These two concepts are closely related, though invariance of a class of games is strictly stronger. Both of these concepts are inherently useful, though for the purpose of our analysis, it is more practical to focus on the invariance of particular equilibria, which can be used to directly show that a class of games is invariant. Even if a class of games is not invariant, there can still be a nonempty set of invariant equilibria for that class. This set of invariant equilibria are in a sense more robust since their status as equilibria cannot be disrupted by minor changes to the specification of the payoffs. This notion of robustness makes invariance a potential equilibrium selection criterion in settings with multiple equilibria. As an example, consider a two player, complete information winner-pay auction in which the first player values the prize at \$1 and the second player at \$2. Regardless of the tie-breaking rule, there always exists an invariant equilibrium in which both players equilibrium bid distributions have full support over $[0, 1]$. However, if the tie-breaking rule specifies that the second player wins with certainty when there is a tie at a bid of \$1, then there exists another equilibrium in which both players bid exactly \$1. This latter equilibrium is inherently fragile, as it requires a very precise specification of the payoffs. As such, the invariant mixed strategy equilibrium may be more compelling as a solution concept.

The conditions for invariance of equilibrium that we present in this paper are matching conditions. They require that, given some strategy profile, players can "match" a particular payoff, meaning that they can obtain the same payoff or greater via the limit of some sequence of deviations.[3] Our main condition, superior payoff matching (henceforth *SPM*), requires that at any given strategy profile, each player can match the highest payoff that they would receive near that strategy profile across all games within that class.[4] That is, each player

[2] As an interpretation, if the set of equilibria of a class of games is invariant to the particular payoff function, then all such games are identical *in equilibrium*. The in equilibrium qualifier is the defining characteristic of this invariance equivalence relation. This is entirely distinct from the notion of strategic equivalence (von Neumann and Morgenstern (1947), and McKinsey (1950)), which specifies that games are strategically equivalent if they have the same best response functions. Strategic equivalence, so named because players have the same strategic incentives in each game, is much stronger than invariance, which places no restrictions off the equilibrium path. Indeed, two games can be invariant even if their payoffs reflect very different applications with correspondingly different payoffs and best response functions, so long as these payoff differences occur off the equilibrium path. Further, the assumption that games within the same class require identical strategy spaces can be relaxed; so long as the strategy spaces for the two games are homeomorphic, there exists a transformation which can place them in the same class. Thus, invariance can be seen as a weaker notion of equivalence between games.

[3] Allison and Lepore (2014) first established the value of matching conditions to verify existence of Nash equilibrium in a discontinuous games by introduction the concept of disjoint payoff matching. He and Yannelis (2016) adapt the same matching condition to the setting of Bayesian game. Allison et al. (2018) also make use of a matching condition as part of sufficient conditions weak reciprocal upper semicontinuity of a games mixed extension.

[4] In studying mixed strategy equilibria of games with strategy spaces in $\mathbb{R}$, Dasgupta and Maskin (1986)

must be able to match the upper semicontinuous envelope of their payoff. While the technical definition is less than intuitive, SPM is a naturally emergent condition in most economic applications. In the context of most economic applications, SPM simply requires that players be able to deviate to obtain their ideal payoffs from points of discontinuities. For example, when the discontinuities are the result of ties, such as in an auction, SPM requires that each player is able to match the payoff they receive if they win any tie with certainty. In most cases, this is trivially achieved via an arbitrarily small increase in the bid.[5] Though it is often the case, as in auctions, that the matching deviations are local, this is not required by SPM. Our main result is that SPM is a sufficient condition for all equilibria of a game to be invariant within a particular sub-class of games.[6] The particular sub-class for which invariance is guaranteed is easily understood, and provides as an immediate corollary a sufficient condition for invariance of the entire class.

An immediate and important application of our invariance results is that they can be used to verify existence of equilibrium in discontinuous games. Not only is this a convenient method for obtaining existence, but our results also cover games which did not satisfy the conditions of previous approaches. As it turns out, SPM, our sufficient condition for invariance of equilibrium within a class of games is also a sufficient condition for existence of equilibrium. More specifically, if the lower semicontinuous envelope of a game's payoffs satisfies SPM and there is a game within the class that satisfies weak reciprocal upper semicontinuity, then that game satisfies better reply security, a condition introduced by Reny (1999) that is sufficient for existence of equilibrium. Our invariance results then imply existence of equilibrium for all games within this associated class. While our proof is built on Reny's conditions, our result is stronger in two dimensions. First, SPM guarantees existence of an invariant Nash equilibrium. Second, because the equilibrium is invariant, this allows for verification of existence of equilibrium for games that do not satisfy Reny's conditions. In particular, we demonstrate via examples that our result can prove existence of pure strategy equilibrium in a game that is not quasiconcave as well as a game that is not better reply secure. Thus, our existence results are a novel contribution to the literature on existence of equilibrium in discontinuous games. We then take a similar approach to provide a new result for existence of a symmetric equilibrium and apply those to verify that a symmetric equilibrium exists in an asymmetric game.

make some assumptions to guarantee that either left or right hand limits in a player's own strategy at points of discontinuity allow the player to deviate locally to obtain the highest possible payoff at that point. In this sense, our condition SPM is a vast generalization in that the deviations need not be local and the payoffs need not be so well behaved at any strategy profile, in addition to the fact that our strategy spaces need not be in $\mathbb{R}$.

[5] SPM is satisfied even in auctions that do not possess pure strategy equilibria. More importantly, these games' mixed extensions also satisfy SPM. The logic and technical requirements are identical in this case.

[6] The condition of SPM in this result can be replaced with a weaker condition which requires the matching be satisfied only at Nash equilibrium strategy profiles. While technically more general, SPM has more practical use as it does not require any equilibrium characterization and has stronger implications for existence of equilibrium.

The invariance and existence results discussed above can be applied to the mixed strategy equilibria of a class of games by considering the direct translation of SPM to the games' mixed extensions. Nevertheless, it is inconvenient to do so for two reasons. First, the upper semicontinuous envelope of a player's payoff function in the mixed extension does not necessarily coincide with the expectation of the upper semicontinuous envelope of her payoff in pure strategies at any particular strategy profile. Second, the weak topology of the mixed extension is computationally difficult to work with and generally ill-suited for use in application. As such, it is desirable to limit analysis to the pure strategy profiles of a game. For that purpose, we introduce an additional sufficient condition for SPM, uniform superior payoff matching (henceforth USPM), which requires that for each strategy in a player's strategy space, that player can match the upper semicontinuous envelope of her payoff against all strategy profiles of the other players with a single sequence of deviations. We then show that USPM is sufficient for the mixed extension to satisfy SPM. Furthermore, if the game satisfies a weak efficiency condition (which is strictly weaker than reciprocal upper semicontinuity), then USPM implies that the game's mixed extension satisfies weak reciprocal upper semicontinuity. A consequence is that USPM is a sufficient condition for existence of mixed strategy Nash equilibrium. This condition is advantageous over other conditions for existence of mixed strategy equilibrium in that it requires no verification in the mixed extension of a game, and thus can be verified intuitively.

Ours is not the first paper to utilize a class of games to characterize a game within that class. Simon and Zame (1990) (henceforth *S&Z*) provide a very general existence of equilibrium result by considering a class of games that is very similar to a game's associated class.[7] In particular, they show that for any compact game, there exists a game within its associated class that possesses a mixed strategy Nash equilibrium.[8] While this result is extremely powerful due to its lack of restrictions, it faces a difficulty for application in that there is no way of knowing which game possesses an equilibrium. The results of this paper naturally pair with S&Z's results since the property of invariance removes this singular drawback. That is, given a game, if all games within its associated class satisfy SPM, then there exists an invariant mixed strategy Nash equilibrium for all such games satisfying our weak efficiency condition.[9]

Carmona and Podczeck (2018) also provide conditions for the invariance of the mixed strategy equilibrium set. They focus on the class of games considered by S&Z and they show that every game with an endogenous sharing rule satisfying virtual continuity and strong indeterminacy has a strategy profile that is an equilibrium for each possible sharing rule.[10]

[7] The class of games that they consider is a subset of a game's associated class as we have defined it.

[8] This result was extended by Jackson et al. (2002) to the setting of incomplete information.

[9] In the process of obtaining this result, we generalize the results of S&Z to allow for strategy spaces to be Hausdorff as opposed to metric. The only nontrivial aspect of this extension is in proving that the strategy spaces can be finitely approximated. The existence of a finite approximation for compact Hausdorff spaces was proved by Kopperman and Wilson (1997).

[10] Intuitively speaking, virtual continuity means that every player can, with probability close to one, avoid

As Carmona and Podczeck note "Verifying virtual continuity in a particular game with an endogenous sharing rule is potentially daunting as one needs to consider all possible selections of the payoff correspondence." Our approach to invariance of mixed strategy equilibrium is different from the approach of Carmona and Podczeck in an important way since we actually study a larger class of games than S&Z, as noted above. This is important for our method of verifying existence of equilibrium, as it allows us to identify the strategy of nesting a game into a class of games that are similar enough to guarantee shared equilibria, even if those games would not be within the class considered by S&Z.

In examining the relationship between SPM and existence of equilibrium using Reny's approach of better reply security and S&Z's approach of an endogenous sharing rule, an interesting connection between these two papers emerges. If all games within a class satisfy SPM, then the sub-class of games satisfying our weak efficiency condition, which is the set of games for which we can guarantee existence of equilibrium using the S&Z approach, is also the set of games within the class for which weak reciprocal upper semicontinuity can be readily verified. Thus, from a practical perspective, the results of Reny and S&Z are not distinct using our approach.[11]

The remainder of the paper is organized as follows. In Section 2 we specify the preliminaries of the model. In Section 3 we present our primary invariance results. In Section 4, we present our primary existence of equilibrium results. In Section 5 we extend all of our results to the mixed extension of the class of games, drawing connections between the approaches of Reny and S&Z. In the final section of the paper, we present an application with less abstract structure to make it clear how our results can be used to extend the analysis of many of the applications of discontinuous games in economics.

2 Preliminaries

Consider a class of games $\mathcal{G}$ with a countable set of players N. We denote this class by $\mathcal{G} = (X,\mathcal{U})$, where $X = \times_{i \in N} X_i$ is the product of each player i's strategy space X_i and $\mathcal{U}$ is a collection of payoff functions $u : X \to \mathbb{R}^N$.[12] A game $G(u) \in \mathcal{G}$ is a pair of the strategy

points where the payoff is multi-valued while virtually achieving the same payoff given the strategies of others, regardless of the particular sharing rule under consideration. Strong indeterminacy requires that indeterminacies are not fully eliminated by focusing on efficient sharing rules.

[11] This is true under the assumptions of our paper but it is certainly not true in general. Carmona and Podczeck (2017) demonstrate that it is not generally the case that the approaches of Reny and S&Z overlap.

[12] Such a selection is referred to as a sharing rule by S&Z, as it specifies how payoffs are shared at points of discontinuity. This is natural in their setting, which coincides with many applications, as all sharing rules agree except at points of discontinuity. However, our setting does not require this property, as we allow the consideration of classes of games which may be fundamentally different, with different payoffs at points of continuity and discontinuity alike. Hence, we do not adopt the term sharing rule, even though a natural application of our results is to a setting akin to the one considered by S&Z where the interpretation of a utility function as a sharing rule is valid.

space along with a payoff function, that is, $G(u) = (X, u)$ for some $u \in \mathcal{U}$. We will suppress the argument and write $G = (X, u)$ when there is no ambiguity as to which payoff function the game uses.

While each payoff function $u \in \mathcal{U}$ represents different preferences over strategy profiles $x \in X$, the player's preferences can be derived from consistent primitives. Let Ω be the set of all possible outcomes that could result from choices in X. Each player i's preferences over the set Ω are represented by a bounded utility function $\phi_i : \Omega \to \mathbb{R}$. The outcome ω that results from each strategy x depends on the game that the players are playing. That is, given any game $G = (X, u)$, there is an associated mapping $\omega_u : X \to \Omega$ that determines the outcome given any strategy profile. The utility functions u_i are then derived from the composition of ϕ_i and ω_u, formally $u_i(x) = \phi_i(\omega_u(x))$ for all $x \in X$. Therefore a game represents a transformation of actions into outcomes, and the preferences of the class of games allows for an unambiguous comparison between the outcomes of games. Note that the boundedness of each ϕ_i implies that the family of functions U is uniformly bounded.[13] Since all relative information for each game $G(u)$ is captured by u, we will henceforth omit all reference to the outcomes and utilities ϕ_i over outcomes except for brief discussions where this derivation is relevant.[14]

Let τ_i denote the topology on X_i and τ the product topology on X. For any $u \in \mathcal{U}$, let $\overline{u} = (\overline{u}_1, \overline{u}_2, ...)$ and $\underline{u} = (\underline{u}_1, \underline{u}_2, ...)$, where $\overline{u}_i$ and $\underline{u}_i$ are the upper and lower semicontinuous envelope of u_i, respectively, defined for each player i as

$$\begin{aligned} \overline{u}_i(x) &= \inf_{V \in \tau : x \in V} \sup_{y \in V} u_i(y) \text{ and} \\ \underline{u}_i(x) &= \sup_{V \in \tau : x \in V} \inf_{y \in V} u_i(y). \end{aligned}$$

Similarly, define the upper and lower bounds across all payoffs in $\mathcal{U}$

$$\begin{aligned} \overline{\pi}_i(x) &= \inf_{V \in \tau : x \in V} \sup_{y \in V} \sup_{u \in \mathcal{U}} \overline{u}_i(y) \text{ and} \\ \underline{\pi}_i(x) &= \sup_{V \in \tau : x \in V} \inf_{y \in V} \inf_{v \in \mathcal{U}} \underline{u}_i(y). \end{aligned}$$

In this way, the functions $\overline{\pi}_i$ and $\underline{\pi}_i$ represent the highest and lowest feasible payoffs across all games within the class.[15] An immediate implication of this definition is that $\overline{\pi}_i$ and $\underline{\pi}_i$ are measurable functions, since they are upper and lower semicontinuous, respectively.

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[13] If necessary, then the utilities φ_i can be scaled so that $|v_i| \leq m$ for all players i for some $m > 0$.

[14] It is entirely possible that there are two different games (X, u) and (X, v) such that $u = v$. This does not imply that the games are identical, only that the players are indifferent between the outcomes of any strategy choices in the two games.

[15] The results of this paper pertaining to pure strategy equilibria could be generalized by redefining these upper and lower bounds as $\sup_{u \in \mathcal{U}} \overline{u}_i(x)$ and $\inf_{v \in \mathcal{U}} \underline{u}_i(x)$, respectively. However, it is easy to construct examples in which these functions are not measurable, which creates problems in the mixed extension. As such, we define the bounds as we have in order to maintain consistency throughout the paper.

Remark 1 *Throughout this paper, our results are written in reference to games with a class $\mathcal{G}$, though it is worth noting that they can apply to other games as well. In particular, the results of this paper can be applied to any game whose payoffs u are bounded between $\underline{\pi}$ and $\overline{\pi}$. The reason is that any such game could simply be added to the class, allowing the results to apply.*

Remark 2 *In application, one typically begins with a game $G(u)$. Any such game belongs to a natural* associated class *of games $\mathcal{G}(u) = (X, \mathcal{U}(u))$, where $\mathcal{U}(u) = \{v : X \to \mathbb{R} : v$ is measurable and $\underline{u} \leq v \leq \overline{u}\}$. Note that for any u, all $v \in \mathcal{U}(u)$ coincide on the set of continuity points of u. Thus, the multitude of payoff possibilities at points of discontinuity in a game's payoff function naturally give rise to a class of possible payoff functions.*

The set of strategies at which the payoffs are not specified to be maximal for a game $G(u) \in \mathcal{G}$ is given by $\Sigma(u) = \{x \in X : u(x) \neq \overline{\pi}(x)\}$. It should be clear that $\Sigma(u)$ need not coincide with $\Sigma(v)$ for all $v \in \mathcal{U}$. Given a payoff function u, let $\mathcal{U}_u = \{v \in \mathcal{U} : \Sigma(v) \subset \Sigma(u)\}$, the subclass of all payoff functions in $\mathcal{U}$ with weakly smaller sets of strategies at which payoffs are not maximal.

We will occasionally make reference to the closure and convex hull of sets. For any set E denote the closure of E by cl$E$ and the convex hull of E by co$E = \{x \in X : x = \alpha y + (1-\alpha)z$ for some $\alpha \in [0,1]$ and $y, z \in E\}$.

Lastly, let $EQ(u)$ denote the set of pure strategy Nash equilibria of a game $G = (X, u)$ and $EQ(\mathcal{U}) = \bigcup_{u \in \mathcal{U}} EQ(u)$. We also denote by $EQ(u, v) = EQ(u) \cap EQ(v)$.

3 Invariant Games

In this section, we formally define the notions of invariant equilibria and invariant games. We then present a simple, intuitive condition under which these notions of invariance are satisfied.

The invariant equilibria for a class of games is the set of strategy profiles that constitute pure strategy Nash equilibria of every game in that class. Formally, the *invariant equilibrium set* of a class of games $\mathcal{G}$ is $IE(\mathcal{U}) = \bigcap_{u \in \mathcal{U}} EQ(u)$. We say that any $x \in IE(\mathcal{U})$ is an *invariant equilibrium.* The invariant equilibria of a class of games are the only equilibria robust to perturbations of the specification of the game within the class. Therefore, one potential use for the concept of equilibrium invariance is as an equilibrium selection criterion.

Two games are invariant if (1) they have the same set of Nash equilibria and (2) the payoffs in any such equilibrium is the same in each game. Formally, $G(u)$ and $G(v)$ are *invariant* if $EQ(u) = EQ(v)$ and $u(x) = v(x)$ for all $x \in EQ(u) = EQ(v)$. A class $\mathcal{G}$ is *invariant* if any two games within the class are invariant. This notion of invariance imposes that payoffs be identical across games *in equilibrium*. As such, invariant games may have payoffs which differ except for at a single strategy profile. The concept of invariance is a weak form of equivalence between games, and indeed the property of invariance defines an equivalence relation in the space of all games with a fixed set of players and strategies.

The requirement that invariant games share equilibrium payoffs derives from a practical interpretation of the games. If invariant games are meant to be equivalent to the players, then they ought to be indifferent between the games. Given the construction of the payoffs, this is only possible if the equilibrium payoffs are identical. Thus, invariance of equilibrium alone is not enough to ensure that games seem equivalent to the players.

Remark 3 *At first glance, it appears that there is an implicit cardinality to the payoff functions that is imposed by invariance. To see this, observe that invariance is not preserved under increasing transformations. Nevertheless, this apparent cardinal requirement is inconsequential, and the fact that invariance is not preserved under such transformations is both natural and intuitive given the derivation of the payoff functions. To see why, suppose that there are payoff functions* $u, v \in \mathcal{U}$ *with* $u_i = f_i(v)$ *for some strictly increasing functions* $f_i : \mathbb{R} \to \mathbb{R}$*. Then the fact that* $u_i(x) = v_i(x)$ *simply means that* $\phi_i(\omega_u(x)) = f_i(\phi_i(\omega_v(x))$*, which is to say that the outcome* $\omega_u(x)$ *has the same utility under* ϕ_i *as the outcome* $\omega_v(x)$ *does under* $f_i(\phi_i)$*, a meaningless statement. Thus, if* $u_i(x) = f_i(v_i(x))$ *and* $f_i(v_i(x)) \neq v_i(x)$*, then the derivation of* u *and* v *implies that player* i *either strictly prefers* $\omega_u(x)$ *to* $\omega_v(x)$ *or vice versa. Thus, such games should not be considered equivalent. In general, the cardinality of a given payoff function* u *is a natural consequence of the preservation of the ordinal preferences of* ϕ*.*

Remark 4 *A trivial consequence of the definition of invariant games is that all games that do not possess a Nash equilibrium are invariant. While one could add an additional requirement that an equilibrium exist, it may at times be natural to consider such games to be equivalent, as they are games without solutions or values.*

The following definition is primitive to the conditions developed in this paper.

Definition 1 *Given a game* $G = (X, u)$*, a player* i *can* match a payoff $\alpha \in \mathbb{R}$ *at* $x \in X$ *if there exists a sequence* $\{x_i^k\} \subset X_i$ *such that* $\liminf_k u_i(x_i^k, x_{-i}) \geq \alpha$*. Equivalently, a player* i *can match a payoff* $\alpha \in \mathbb{R}$ *at* $x \in X$ *if for any* $\varepsilon > 0$*, there exists an* $x_i' \in X$ *such that* $u_i(x_i', x_{-i}) \geq \alpha - \varepsilon$*.*

A player can match a payoff α at a strategy profile if that player can deviate and receive a payoff that is either greater than α or arbitrarily close. We find the latter equivalent statement to be less useful for the results of this paper, however, it does serve to highlight a couple facts. First, it makes clear that a single deviation is sufficient to match a payoff of α if that deviation yields a payoff weakly greater than α. Second and more important, being able to match a payoff is strictly (and significantly) weaker than being able to secure a payoff.[16] The fact that matching a payoff does not impose any conditions on the deviation payoff in a neighborhood of the other players' strategies makes it much easier to verify than security concepts, particularly when applied to the mixed extension of games.

The following payoff matching condition imposes restrictions on a given game G relative to the highest possible payoff within the class of games $\mathcal{G}$.

Definition 2 *A game $G \in \mathcal{G}$ satisfies* superior payoff matching (SPM) *if for each player $i \in N$ and all $x \in X$ there exists a sequence $\{x_i^k\} \subset X_i$ such that* $\liminf_k u_i(x_i^k, x_{-i}) \geq \overline{\pi}_i(x)$.

A game satisfies superior payoff matching if given any strategy profile, any player can deviate to do at least as well as the highest feasible payoff at that strategy profile. If every game $G \in \mathcal{G}$ satisfies SPM, then we say that the class of games $\mathcal{G}$ satisfies SPM.

Note that the sequence of deviations specified by SPM need not converge to the original strategy profile. It should be clear that SPM is trivially satisfied if $u = \overline{\pi}$. In particular, this implies that for a given game $G(u)$ in the singleton class $\{G(u)\}$, SPM if satisfied if u is upper semicontinuous, though the converse of this fact is false since the deviations need not be local.

Our first major result is to show that SPM is a sufficient condition for invariance within a class of games.

Theorem 1 *Suppose that $G(u)$ satisfies SPM. Then $EQ(u) = IE(\mathcal{U}_u)$ and $u(x) = v(x) = \overline{\pi}(x)$ for all $u \in \mathcal{U}$, all $v \in \mathcal{U}_u$ and all $x \in EQ(u)$. Consequently, if G satisfies SPM for all $G \in \mathcal{G}$ and $\Sigma(u) = \Sigma(v)$ for all $u, v \in \mathcal{U}$, then $\mathcal{G}$ is an invariant class.*

Theorem 1 demonstrates that SPM is a sufficient condition both for equilibria to be invariant as well as games. The necessity of the restriction to $\mathcal{U}_u$ in this result is obvious, as otherwise one could simply consider a v for which $v(x) < \overline{\pi}(x)$ for some equilibrium $x \in EQ(u)$ and observe that invariance of equilibrium is violated.

Proof of Theorem 1. Suppose that $G(u)$ satisfies SPM and let $x \in EQ(u)$ and $v \in \mathcal{U}_u$. Note first that $u(x) = \overline{\pi}(x)$. Otherwise, if $u_i(x) < \overline{\pi}_i(x)$ for some player i, then SPM guarantees that there exists an $x'_i \in X_i$ such that $u_i(x'_i, x_{-i}) > u_i(x)$. It follows that $v(x) = \overline{\pi}(x)$ since $\Sigma(v) \subset \Sigma(u)$. Suppose that $x \notin EQ(v)$. Then there exists a player $i \in N$ with strategy $x'_i \in X_i$ such that $v_i(x'_i, x_{-i}) > v_i(x) = \overline{\pi}_i(x)$. From SPM, let x_i^k be such that $\liminf_k u_i(x_i^k, x_{-i}) \geq \overline{\pi}_i(x'_i, x_{-i})$. Then since $\overline{\pi}_i(x'_i, x_{-i}) \geq v_i(x'_i, x_{-i}) > \overline{\pi}_i(x) = u_i(x)$, it

[16] A player can secure a payoff of α at x if for all $\varepsilon > 0$, there exists a deviation $x'_i \in X$ and neighborhood V of x_{-i} such that $u_i(x'_i, x'_{-i}) \geq \alpha - \varepsilon$ for all $x'_{-i} \in V$.

must be that $u_i(x_i^k, x_{-i}) > u_i(x)$ for sufficiently large k, violating x as an equilibrium. We conclude that $x \in EQ(v)$.

The latter statement follows immediately from the former. ■

A potential concern for verifying that a class of games is invariant using SPM is that this requires verification of SPM for each game within the class. The following theorem alleviates this potential burden, and offers the potential to verify SPM for the whole class using only a single payoff function.

Theorem 2 *Let $\mathcal{G}$ be a class of games. If $G(u)$ satisfies SPM for some $u \in \mathcal{U}$, then $G(v)$ satisfies SPM for all functions v with $u \leq v \leq \overline{\pi}$. In particular, if $G(\underline{\pi})$ satisfies SPM, then $\mathcal{G}$ satisfies SPM.*

While the invariance of $\mathcal{G}$ does not necessitate that $G(\underline{\pi})$ satisfies SPM, this is a convenient sufficient condition in practice. The necessary condition for the invariance of $\mathcal{G}$ is that all games whose payoffs are minimal elements of $\mathcal{U}$ under the standard partial order satisfy SPM, though this property is intrinsically less practical for application.

Proof of Theorem 2. Let $u \in \mathcal{U}$ be such that $G(u)$ satisfies SPM and let v be a function with $u \leq v \leq \overline{\pi}$. Let $x \in X$ and $i \in N$. From SPM, let x_i^k be such that $\liminf_k u_i(x_i^k, x_{-i}) \geq \overline{\pi}_i(x_i, x_{-i})$. Then since $v \geq u$, it follows immediately that $\liminf_k v_i(x_i^k, x_{-i}) \geq \overline{\pi}_i(x_i, x_{-i})$. We conclude that $G(v)$ satisfies SPM.

The latter result follows immediately from the fact that $u \geq \underline{\pi}$ for all $u \in \mathcal{U}$. ■

Corollary 1 *Let $\mathcal{G}$ be a class of games. If $G(\underline{\pi})$ satisfies SPM and $\Sigma(u) = \Sigma(v)$ for some all $u, v \in \mathcal{G}$, then $IE(\mathcal{U}) = EQ(\underline{\pi})$.*

Corollary 1 follows immediately from Theorems 1 and 2.

As mentioned above, invariance defines an equivalence relation $\sim$ on $\mathcal{U}$ defined by $u \sim v$ if and only if $EQ(u) = EQ(v)$ and $u(x) = v(x)$ for all $x \in EQ(u)$. Given the results above, SPM can be used to characterize this equivalence relation via easily observed properties of the utility functions.

Theorem 3 *Let $\mathcal{G}$ be a class of games. If $\mathcal{G}$ satisfies SPM, then the invariance equivalence relation $\sim$ is defined by $u \sim v$ if and only if $\Sigma(u) = \Sigma(v)$.*

Now we move to establishing the connection between invariance and existence of equilibrium.

4 Existence of Pure Strategy Equilibrium

In this section, we show that SPM is useful for proving existence of equilibrium, as it provides an easily verifiable sufficient condition that is relatively general and guarantees existence of an invariant equilibrium for all games in the same class that have a weakly smaller set of non-maximal payoffs. Our existence results do require that at least one game in the class satisfies quasi-concavity and weak reciprocal upper semicontinuity in addition to SPM, however, the same need not be true of all games in the class. We are therefore able to use this approach to prove existence of equilibrium in some games whose payoffs are not quasiconcave and/or do not satisfy weak reciprocal upper semicontinuity.

For the remainder of this section, we assume that X is a compact, convex subset of a topological vector space and that the set of player is finite, where we write $N = \{1, 2, ..., n\}$.

We use the following conditions and results of Reny (1999) and Bagh and Jofre (2006) to obtain existence of an equilibrium for a particular payoff function. In order to present Reny's conditions, we must first present the notion of securing a payoff. A player $i \in N$ can *secure* a payoff of α at a strategy profile $x \in X$ if there exists an $x_i' \in X_i$ and neighborhood V of x_{-i} such that $u_i(x_i', x_{-i}') \geq \alpha$ for all $x_{-i}' \in V$.

Definition 3 *A game G is* better reply secure *if whenever $(x^*, u^*) \in clG$ and x^* is not an equilibrium, then there is some player i that can secure a payoff strictly higher than u_i^* at x^*.*

Fact 1 (Reny (1999)) *If G is better reply secure and quasiconcave, then G possesses a pure strategy Nash equilibrium.*

Reny further provides two conditions that together are sufficient for a game to be better reply secure.

Definition 4 *A game G is* payoff secure *if for every $x \in X$ and every $\varepsilon > 0$, each player i can secure a payoff of $u_i(x) - \varepsilon$ at x.*

Definition 5 *A game G satisfies* reciprocal upper semicontinuity (RUSC) *if whenever $(x^*, u^*) \in clG$ and $u(x^*) \leq u^*$, then $u(x^*) = u^*$.*

Fact 2 (Reny (1999)) *If G satisfies payoff security and RUSC, then it is better reply secure.*

Bagh and Jofre (2006) introduced the follow weakening of RUSC that can replace RUSC in the previous result.

Definition 6 *The game G satisfies* weak reciprocal upper semicontinuity (WRUSC) *if whenever $(x^*, u^*) \in clG \setminus G$, there is some player $i \in N$ and $x_i \in X_i$ such that $u_i(x_i, x^*_{-i}) > u^*_i$.*

Fact 3 (Bagh and Jofre (2006)) *If G satisfies payoff security and WRUSC, then it is better reply secure.*

Our main existence result relates SPM to payoff security, and combines this with our invariance theorem and Bagh and Jofre's result to obtain existence of an invariant pure strategy Nash equilibrium.

Theorem 4 *Consider a class of games $\mathcal{G}$, and $G(u) \in \mathcal{G}$. If $G(\underline{u})$ satisfies SPM and $G(u)$ satisfies WRUSC and quasi-concavity, then there exists an invariant pure strategy Nash equilibrium $x^* \in IE(\mathcal{U}_u)$.*

Corollary 2 *If a game $G(u)$ satisfies WRUSC and quasiconcavity, and its associated class $\mathcal{G}(u)$ satisfies SPM, then there exists an invariant pure strategy equilibrium $x^* \in IE(\mathcal{U}(u))$, and thus $EQ(u) \neq \emptyset$.*

To prove Theorem 4, we need only show that $G(u)$ is payoff secure, which we demonstrate in Lemma 1 below. Then, Fact 3 guarantees that $G(u)$ is better reply secure, thereby satisfying the conditions of Fact 1, which guarantees the existence of a pure strategy equilibrium x^* for $G(u)$. Finally, we apply Theorem 1, which guarantees that any equilibrium is also an equilibrium for all $v \in \mathcal{U}_u$.

Again, it is important to note that the existence result of Theorem 4 is not a special case of Reny's results. While we do use better reply security and quasi-concavity to find an equilibrium for the game with payoff function u, the other payoff functions $v \in \mathcal{U}_u$ need not satisfy quasiconcavity, WRUSC, or better reply security. Therefore, Theorem 4 is a new pure strategy Nash equilibrium existence result. This is formally demonstrated by the following examples.

Remark 5 *It is worth taking a moment to discuss the strategy by which we will prove existence of equilibrium in these examples, as this is a viable strategy for the application of our results. For each of these examples, we are faced with a game that does not satisfy the properties that we need to verify existence. This is a perfect setting to apply our invariance results, as we do not need any game in particular to be well behaved, as long as it shares equilibria with a game that we are able to verify existence. In order to do this, we embed the game into a class of games which satisfies SPM, then use the implied invariance to extend the equilibria of games within that class to the game of interest. Finding the right class of games may be as simple as considering the game's associated class, or it may require some clever observation or construction. In general, it is only necessary to group the game's utility function with a single other utility function that satisfies desirable properties.*

This first example demonstrates that Theorem 4 can be used to verify existence of equilibrium in games that do not satisfy quasiconcavity.

Example 1 *Let* $X_1 = X_2 = [0,1]$, $N = \{1,2\}$ *and* u *defined by*

$$u_i(x) = \begin{cases} 2(1 - |x_i - 3/(3 + x_{-i})|) & \text{if } x_i > 1/2 \\ 7/4 & \text{if } x_i = 1/2 \\ x_i & \text{if } x_i < 1/2 \end{cases}.$$

Note that u_i *does not satisfy quasiconcavity around the point* $x_i = 1/2$.

Consider the game $G(u)$ *and its associated class* $\mathcal{U}(u)$. *Note that* $G(\underline{u})$ *satisfies SPM with the sequence of deviations* $x_i^k = 3/(3 + x_{-i})$ *for all* k, *for any* $x \in X$ *and is quasi-concave since* $\underline{u}_i(1/2, x_{-i}) = 1/2$. *The payoff is strictly increasing up to* x_i^k *and strictly decreasing thereafter. Moreover, since* $\underline{u}_i$ *is continuous in* x_{-i} *and continuous in* x_i *except at* $x_i = 1/2$ *and* $\underline{u}_i(x_i^k, x_{-i}) = 2 > 7/4 = \overline{\pi}_i(1/2, x_{-i})$ *for all* x_{-i}, *then* $G(\underline{u})$ *is WRUSC. Therefore, Theorem 4 guarantees that there exists a pure strategy Nash equilibrium* $x^* \in IE(\mathcal{U}_{\underline{u}}) = IE(\mathcal{U}(u))$. *In particular, this guarantees that* $G(u)$ *has a pure strategy Nash equilibrium.*

The next two examples demonstrate that Theorem 4 can be used to verify existence of equilibrium in games that do not satisfy better reply security. In the first of these, WRUSC is violated, while in the latter, payoff security is violated.

Example 2 *Let* $X_1 = X_2 = [0,2]$, $N = \{1,2\}$ *and* u, v *defined by*

$$\begin{aligned} u_i(x) &= \begin{cases} x_i & \text{if } x_i \leq 2 - x_{-i} \\ 2 & o.w. \end{cases}, \\ v_i(x) &= x_i. \end{aligned}$$

Note that $G(u)$ *does not satisfy better reply security since* $((1,1),(2,2)) \in clG(u)$ *and* $(1,1)$ *is not a Nash equilibrium, but no player can secure or obtain a payoff higher than* 2.

Consider the class of games with $\mathcal{U} = \{u, v\}$, *noting that* $\underline{\pi} = v$, *so* $\Sigma(u) \subset \Sigma(v)$. *Note that* $G(v)$ *satisfies SPM with any sequence of deviations* $x_i^k \to 2$ *for any* $x \in X$. *Moreover, since each* v_i *is continuous and is monotonic in* x_i, $G(v)$ *is quasiconcave and satisfies WRUSC. Therefore, Theorem 4 guarantees that there exists a pure strategy Nash equilibrium* $x^* \in IE(\mathcal{U}_v) = IE(\mathcal{U})$. *In particular, this guarantees that* $G(u)$ *has a pure strategy Nash equilibrium.*

Example 3 *Let* $X_1 = X_2 = [0,1]$, $N = \{1,2\}$, *and* u, v, w *defined by*

$$u_i(x) = \begin{cases} 0 & \text{if } x_{-i} \in \mathbb{Q} \diagdown \{1\} \\ x_i & o.w. \end{cases},$$
$$v_i(x_i) = x_i.$$
$$w_i(x_i) = \frac{1}{2} + \frac{x_i}{2}.$$

Note first that neither player can secure any payoff greater than zero at any strategy profile in $G(u)$. *The reason is that there are rationals in every neighborhood of the other player's strategy, and thus payoffs are zero at some strategies of every neighborhood each strategy profile. Consequently, the game is neither payoff secure nor better reply secure.*

Consider the class of games $\mathcal{U} = \{u, v, w\}$. *Then note that* $\overline{\pi}_i = w_i$ *for each player* i, *and* $\Sigma(u) = \Sigma(v) = X \diagdown \{(1,1)\}$. *Thus,* $u \in \mathcal{U}_v$. *Furthermore,* v_i *is continuous, so* $v_i = \underline{v}_i$ *and* $G(v)$ *satisfies WRUSC. Next,* $G(v)$ *satisfies SPM with the sequence* $x_i^k = 1$ *for each player* i *and all* k. *Lastly, since* v_i *is monotonic, it is quasiconcave. Therefore, Theorem 4 guarantees that* $IE(\mathcal{U}_v)$ *is nonempty, and in particular,* $EQ(u)$ *is nonempty.*

The previous two examples highlight both the novelty of Theorem 4 as well as some useful techniques in implementing this theorem to verify existence of equilibrium in a given discontinuous game. In Example 2, the inclusion of a well behaved payoff function that lies below a problematic function, but is constructed to share equilibria allows for verification of existence. In a very different approach, Example 3 includes a well behaved payoff function that ought to share equilibria as well as some larger payoff function to increase the set of nonmaximal payoffs of the well behaved payoff function so that they contain those of the problematic function, allowing application of Theorem 3.

Theorem 4 is proved using the following lemma, which shows that if $G(\underline{u})$ satisfies SPM, then $G(u)$ must be payoff secure.

Lemma 1 *Let* $\mathcal{G} = (X, \mathcal{U})$ *be a class of games and let* $u \in \mathcal{U}$. *If* $G(\underline{u})$ *satisfies SPM, then for all* $x^* \in X$ *and all* $\varepsilon > 0$, *each player* i *can secure a payoff of at least* $\sup_{x_i \in X_i} \overline{\pi}_i(x_i, x_{-i}^*) - \varepsilon$ *at* x^* *in the game* $G(u)$. *Consequently,* $G(u)$ *is payoff secure.*[17]

Proof of Lemma 1. Let $u \in \mathcal{U}$ and $x^* \in X$. For all $\varepsilon > 0$, let $x_i^\varepsilon \in X_i$ be such that $\overline{\pi}(x_i^\varepsilon, x_{-i}^*) \geq \sup_{x_i \in X_i} \overline{\pi}(x_i, x_{-i}^*) - \varepsilon/3$. Since $\underline{u}$ satisfies SPM, for any x_i^ε, there exists a sequence x_i^k such that $\liminf_k \underline{u}_i(x_i^k, x_{-i}^*) \geq \overline{\pi}_i(x_i^\varepsilon, x_{-i}^*)$. Thus, there exists a $K(\varepsilon)$ such that $\underline{u}_i(x_i^k, x_{-i}^*) \geq \overline{\pi}_i(x_i^\varepsilon, x_{-i}^*) - \varepsilon/3$ for all $k > K(\varepsilon)$. Let $k > K(\varepsilon)$ and define $\widetilde{x}_i^\varepsilon = x_i^k$,

[17] We could easily obtain another result guaranteeing payoff security for all $G \in (X, \mathcal{U}_u)$ provided that $G(\underline{v})$ satisfies SPM for all $v \in \mathcal{U}_u$. Example 3 demonstrates that it is not sufficient that $G(\underline{u})$ satisfies SPM, since it is possible for $v_i(x) < \underline{u}_i$ for some $v \in \mathcal{U}_u$ and $x \in \Sigma(u)$.

then note that $\underline{u}_i(x_i^k, x_{-i}^*) \geq \sup_{x_i \in X_i} \overline{\pi}_i(x_i, x_{-i}^*) - 2\varepsilon/3$. By construction, $\underline{u}_i(x_i, x_{-i})$ is lower semicontinuous, so there exists a neighborhood V of x_{-i}^* such that $\underline{u}_i(\widetilde{x}_i^\varepsilon, x_{-i}) \geq \underline{u}_i(\widetilde{x}_i^\varepsilon, x_{-i}^*) - \varepsilon/3 \geq \sup_{x_i \in X_i} \overline{\pi}_i(x_i, x_{-i}^*) - \varepsilon$ for all $x_{-i} \in V$. Finally, note that $u_i(x) \geq \underline{u}_i(x)$ for all $x \in X$, so $u_i(\widetilde{x}_i^\varepsilon, x_{-i}) \geq \sup_{x_i \in X_i} \overline{\pi}_i(x_i, x_{-i}^*) - \varepsilon$. Therefore, for all $\varepsilon > 0$, any player $i \in N$ can secure a payoff of $\sup_{x_i \in X_i} \overline{\pi}_i(x_i, x_{-i}^*)$ at x^* in $G(u)$. ■

We now proceed with the proof of Theorem 4.

Proof of Theorem 4. $G(u)$ satisfies WRUSC, and Lemma 1 implies that $G(u)$ satisfies payoff security. Therefore, based on Proposition 1 in Bagh and Jofre (2006), $G(u)$ satisfies better reply security. If, in addition, u is quasi-concave, then Theorem 3.1 of Reny (1999) implies that $G(u)$ possesses a pure strategy Nash equilibrium x^*. It follows from Theorem 1 that $x^* \in IE(\mathcal{U}_u)$. ■

4.1 Existence of Symmetric Pure Strategy Equilibrium

In this section we show that a weakened version of SPM to the diagonal can be used to obtain new results that guarantee the existence of invariant symmetric pure strategy Nash equilibrium. By looking at classes of games and applying our invariance results, we are able to demonstrate existence of symmetric Nash equilibria in games which do not satisfy symmetry assumptions. As we again will utilize the results of Reny (1999), we will use a similar symmetry framework. Since all of these conditions are analogues to the general conditions that are restricted to the diagonal, the proofs in this section are largely identical to those in the previous section.

For this entire subsection, we consider games with symmetric strategy spaces such that $X_1 = ... = X_n = \chi$. We denote the class of games with symmetric strategy spaces and an arbitrary class of payoffs $\mathcal{U}$, by $\mathcal{G}^s = (\chi^n, \mathcal{U})$.

Definition 7 *A game $G \in \mathcal{G}^s$ is* quasi-symmetric *if $u_1(x, y, ..., y) = u_2(y, x, y, ..., y) = ... = u_n(y, ..., y, x)$ for all $x, y \in \chi$.*

Definition 8 *The game $G \in \mathcal{G}^s$ is* diagonally quasi-concave *if χ is convex and for every player i, all $x^1, ..., x^m \in X$, and all $\overline{x} \in co\{x^1, ..., x^m\}$*

$$u_i(\overline{x}, ..., \overline{x}) \geq \min_{1 \leq k \leq m} u_i(\overline{x}, ..., x^k, ..., \overline{x}).$$

Any game (X, u) satisfying the quasi-symmetry assumption has a unique diagonal payoff function $\delta : \chi \to \mathbb{R}$ such that each $u_i(x, ..., x) = \delta(x)$ for all $x \in \chi$. Reny (1999) defines the following diagonal analogues to better reply security and payoff security. A player i can secure a payoff of $\alpha \in \mathbb{R}$ along the diagonal at $(x, ..., x)$ if there exists an $\overline{x} \in \chi$ such that $u_i(x', ..., \overline{x}, ..., x') \geq \alpha$ for all x' in some neighborhood of $x \in \chi$.

Definition 9 *A quasi-symmetric game G satisfies* diagonal better reply security *if whenever (x^*, u^*) is in the closure of the graph of δ and $(x^*, ..., x^*)$ is not an equilibrium, some player i can secure a payoff strictly higher than u^* along the diagonal at $(x^*, ..., x^*)$.*

The following result from Reny (1999) relates diagonal better reply security to the existence of a symmetric equilibrium in quasi-symmetric games.

Fact 4 (Reny (1999)) *If $G \in \mathcal{G}^s$ is quasi-symmetric, compact, diagonally quasi-concave, and diagonally better reply secure, then it possesses a symmetric pure strategy Nash equilibrium.*

Next we introduce the concept of diagonal superior payoff matching (*DSPM*), which is just SPM defined only on the diagonal.

Definition 10 *A game $G \in \mathcal{G}^s$ satisfies* diagonal superior payoff matching (DSPM) *if each player $i \in N$ can match a payoff of $\overline{\pi}_i(x_i, x, ..., x)$ at $(x_i, x, ..., x)$ for each $x_i, x \in \chi$.*

Note that DSPM is strictly weaker than SPM. It is clear that the analogous result to Theorem 2 is true here, that if $G(u)$ satisfies DSPM and $u \leq v \leq \overline{\pi}$, then $G(v)$ satisfies DSPM.

We also require a weaker notion of WRUSC only on the diagonal. This condition is strictly weaker than upper semicontinuity of δ. Define $Diag(X) = \{(x, ..., x) : x \in \chi\}$, and let $(Diag(X), u)$ denote the graph of u on $Diag(X)$.It should be clear that

Definition 11 *A game $G \in \mathcal{G}^s$ satisfies* WRUSC along the diagonal *if whenever $(x^*, u^*) \in cl(Diag(X), u) \diagdown (Diag(X), u)$, there exists for some player $i \in N$ a strategy $x' \in X$ such that $u_i(x', x, ..., x) > u_i^*$.*

The following is our main result pertaining to existence of symmetric pure strategy equilibrium.

Theorem 5 *Let $(\chi^n, \mathcal{U})$ be a class of games. If $G(\underline{u})$ satisfies DSPM and $G(u)$ satisfies WRUSC along the diagonal, quasi-symmetry and diagonal quasi-concavity for some $u \in \mathcal{U}$, then there exists a symmetric pure strategy Nash equilibrium $x^* \in IE(\mathcal{U}_u)$.*

The proof of Theorem 5 does not follow immediately from Theorem 4 because the symmetric version of Proposition 1 in Bagh and Jofre (2006) is not true in general. Nevertheless, our method of proof is identical to that of 4.

Proof of Theorem 5. Suppose that $G(\underline{u})$ satisfies DSPM and that $G(u)$ satisfies WRUSC along the diagonal, quasi-symmetry, and diagonal quasi-concavity. Let $x \in Diag(X)$ be such that $(x^*, u^*) \in \text{cl}(Diag(X), u)$ with $x^* \notin EQ(u)$. Since each $\overline{\pi}_i$ is upper semicontinuous and $u \leq \overline{\pi}$, it follows that $u^* \leq \overline{\pi}(x^*)$. We need to show that there is some player i that can secure a payoff strictly higher than u_i^*. We first show that there is some player i with a strategy x_i' such that $\overline{\pi}_i(x_i', x_{-i}^*) > u_i^*$. If $u_i^* < \overline{\pi}_i(x^*)$, then we are done. Suppose that $u^* = \overline{\pi}(x^*)$. If $u(x^*) \neq \overline{\pi}(x^*)$, then WRUSC along the diagonal guarantees that some player i possesses a strategy $x_i \in \chi$ such that $u_i(x_i, x_{-i}^*) > u_i^*$, and in particular, $\overline{\pi}_i(x_i, x_{-i}^*) > u_i^*$. Else, if $u(x^*) = \overline{\pi}(x^*)$, then the fact that $x^* \notin EQ(u)$ guarantees that some player i has a strategy $x_i \in X_i$ such that $u_i(x_i, x_{-i}^*) > u_i^*$, and thus $\overline{\pi}_i(x_i, x_{-i}^*) > u_i^*$. Therefore, if there is some player i with strategy x_i' such that $\overline{\pi}_i(x_i', x_{-i}^*) > u_i^*$.

Next, since $G(\underline{u})$ satisfies DSPM, for any $\varepsilon > 0$, there exists for player i a strategy $x_i^\varepsilon \in \chi$ such that $\underline{u}_i(x_i^\varepsilon, x_{-i}^*) > \overline{\pi}_i(x_i', x_{-i}^*) - \varepsilon$. By choosing ε sufficiently small, we have $\underline{u}_i(x_i^\varepsilon, x_{-i}^*) > u_i^*$. Since $\underline{u}_i$ is lower semicontinuous, this implies that there exists a neighborhood V of x_{-i}^* such that $\underline{u}_i(x_i^\varepsilon, x_{-i}') > u_i^*$ for all $x_{-i}' \in V$. Finally, since $\underline{u}_i \leq u_i$, it follows that $u_i(x_i^\varepsilon, x_{-i}') > u_i^*$ for all $x_{-i}' \in V$. We conclude that that $G(u)$ satisfies better reply security along the diagonal.

Theorem 4.1 of Reny (1999) therefore guarantees that $G(u)$ possesses a symmetric pure strategy Nash equilibrium x^*. Finally, Theorem 1 guarantees that $x^* \in IE(\mathcal{U}_u)$. ■

The following example considers a class of games with dissimilar payoff functions that do not individually satisfy quasi-symmetry, but for which $G(\underline{\pi})$ satisfies DSPM, WRUSC along the diagonal, quasi-symmetry, and diagonal quasiconcavity. Thus, the example illustrates that Theorem 5 can be used to obtain existence of a symmetric equilibrium in asymmetric games.

Example 4 *Let $X = [0, 1]$ and $N = \{1, 2\}$. Define u and v by*

$$
\begin{aligned}
u_1(x) &= \begin{cases} x_1 + x_2 & \text{if } \; x_1 < 1/2 \\ 1/2 - (x_1 - 1/2)/3 + x_2 & \text{if } \; x_1 \geq 1/2 \end{cases} \\
u_2(x) &= \begin{cases} 1/3 + x_2/3 + x_1 & \text{if } \; x_2 < 1/2 \\ 1/2 - (x_2 - 1/2) + x_1 & \text{if } \; x_2 \geq 1/2 \end{cases} \\
v_1(x) &= \begin{cases} 1/3 + x_1/3 + x_2 & \text{if } \; x_1 < 1/2 \\ 1/2 - (x_2 - 1/2) + x_2 & \text{if } \; x_1 \geq 1/2 \end{cases} \\
v_2(x) &= \begin{cases} x_2 + x_1 & \text{if } \; x_2 < 1/2 \\ 1/2 - (x_2 - 1/2)/3 + x_1 & \text{if } \; x_2 \geq 1/2 \end{cases} .
\end{aligned}
$$

These payoffs are very simple. Each u_i and v_i is piecewise linear and strictly quasi-concave in x_i with the maximizer $x_i = 1/2$. The function u_1 has a slope of 1 on $[0, 1/2)$ and $-1/3$ on $(1/2, 1]$, while u_2 has slope $1/3$ on $[0, 1/2)$ and -1 on $(1/2, 1]$. The payoff function v is identical to u except with the roles of players 1 and 2 reversed.

Define f and g by

$$\begin{aligned} f_i(x) &= 1/2 - |x_i - 1/2|/3 + x_{-i} \\ g_i(x) &= 1/2 - |x_i - 1/2| + x_{-i} \end{aligned}$$

and consider the class of payoffs $\mathcal{U} = \{u, v, f, g\}$. Then notice that $f = \overline{\pi}$ and $g = \underline{\pi}$.

Note that (X, g) is quasi-symmetric, quasi-concave, and satisfies DSPM with the sequence $x_i^k = 1/2$ for any $x_i \in X_i$. Further, WRUSC (and better reply security) is guaranteed by the continuity of g. Thus, Theorem 5 guarantees that a symmetric pure strategy equilibrium exists for all $u \in \mathcal{U}$. However, it is clearly the case that neither u nor v is quasi-symmetric and thus does not satisfy Theorem 4.1 of Reny (1999).

As a final note, the examples of the previous section are all symmetric and can easily shown to satisfy DSPM, and thus also serve as examples for which we can obtain novel existence of equilibrium results.

5 Mixed Strategy Equilibria

In this section, we extend the results of the previous sections to the case of mixed strategy equilibria. For this purpose, we must assume that each u_i is a measurable function. We begin by defining specific notation for the mixed extension of a game. Denote by M_i the set of regular probability distributions on X_i and let $M = \times_{i \in N} M_i$. For any $u \in \mathcal{U}$ and $\mu \in M$, we will use $U(\mu)$ to denote the expectation of $u(x)$ given μ. That is, $U(\mu) = \int u(x) d\mu$. Denote the mixed extension of a game $G(u)$ by $\widetilde{G}(u) = (M, U)$ and the mixed extension of a class of games $\mathcal{G}$ by $\widetilde{\mathcal{G}} = (M, \mathcal{U})$. We also extend the definitions of the functions $\overline{u}_i$, $\underline{u}_i$, $\overline{\pi}_i$, and $\underline{\pi}_i$ to M as follows. Let T_i denote the topology on M_i and T the product topology on M. For any $u \in \mathcal{U}$, let $\overline{U} = (\overline{U}_1, \overline{U}_2, ...)$ and $\underline{U} = (\underline{U}_1, \underline{U}_2, ...)$, where $\overline{U}_i$ and $\underline{U}_i$ are the upper and lower semicontinuous envelope of U_i, respectively, defined for each player i as

$$\begin{aligned} \overline{U}_i(\mu) &= \inf_{V \in T : \mu \in V} \sup_{\lambda \in V} U_i(\lambda) \text{ and} \\ \underline{U}_i(\mu) &= \sup_{V \in T : \mu \in V} \inf_{\lambda \in V} U_i(\lambda). \end{aligned}$$

Similarly, define the upper and lower bounds across all payoffs in $\mathcal{U}$

$$\begin{aligned} \overline{\Pi}_i(\mu) &= \inf_{V \in T : \mu \in V} \sup_{\lambda \in V} \sup_{u \in \mathcal{U}} \overline{U}_i(\lambda) \text{ and} \\ \underline{\Pi}_i(\mu) &= \sup_{V \in T : \mu \in V} \inf_{\lambda \in V} \inf_{v \in \mathcal{U}} \underline{U}_i(\lambda). \end{aligned}$$

While it is not immediately obvious, it need not be the case that $\overline{\Pi}(\mu) = \int \overline{\pi}(x) d\mu$. Example 6, located in the Appendix, formally demonstrates this fact.

We will use $\widetilde{EQ}(u)$ to denote the set of Mixed Strategy Nash equilibria of a game $\widetilde{G}(u)$ and $\widetilde{EQ}(\mathcal{U}) = \bigcup_{u\in\mathcal{U}} \widetilde{EQ}(u)$. Finally, we denote by $\widetilde{IE}(\mathcal{U}) = \bigcap_{u\in\mathcal{U}} \widetilde{EQ}(u)$ the set invariant equilibria across all $u \in \mathcal{U}$. Define the set of mixed strategies with nonmaximal payoffs for $u \in \mathcal{U}$ by $\widetilde{\Sigma}(u) = \{\mu \in M : u(\mu) \neq \overline{\pi}(\mu)\}$.

5.1 Invariant Games

In this section, we present the corollaries of the invariance results of Section 3. While the results themselves extend straightforwardly for the most part, there are a few extensions that are nontrivial. The first such extension that we will discuss is the set of strategy profiles for which payoffs are non-maximal, $\widetilde{\Sigma}(u)$. While this set is very difficult to compute, it is only necessary to compare these sets for different payoff functions. The following lemma demonstrates that it is sufficient to compare the sets $\Sigma(u)$ for the respective payoff functions, thereby allowing some of the analysis to be restricted to the payoffs for pure strategies.

Lemma 2 *For all $u, v \in \mathcal{U}$, $\Sigma(v) \subset \Sigma(u)$ if and only if $\widetilde{\Sigma}(v) \subset \widetilde{\Sigma}(u)$.*

Proof of Lemma 2. The fact that $\widetilde{\Sigma}(v) \subset \widetilde{\Sigma}(u)$ implies $\Sigma(v) \subset \Sigma(u)$ is obvious since pure strategies are degenerate mixed strategies. It remains to show that $\Sigma(v) \subset \Sigma(u)$ implies $\widetilde{\Sigma}(v) \subset \widetilde{\Sigma}(u)$. Suppose that $\mu \in \widetilde{\Sigma}(v)$, so that $v(\mu) \neq \overline{\pi}(\mu)$. Then it must be that $\mu(\Sigma(v)) > 0$. Since $\Sigma(v) \subset \Sigma(u)$, $\mu(\Sigma(u)) \geq \mu(\Sigma(v)) > 0$. This guarantees that $u(\mu) < \overline{\pi}(\mu)$, and thus that $\mu \in \widetilde{\Sigma}(u)$. ■

Based on Lemma 2, all references to sets of strategies at which payoffs are non-maximal are made with respect to the pure strategy sets $\Sigma(u)$.

The next challenge in applying the results of Section 3 to mixed strategy equilibria is that it is generally less tractable to make computations and verify conditions in the mixed extension of games. We will first present the mixed analogues of the results of Section 3, and then in a later subsection we will present some sufficient conditions which can greatly alleviate the burden of verifying the conditions of these results. The definition of superior payoff matching in the mixed extension is obtained by applying the prior definition of SPM to the class of games $\widetilde{\mathcal{G}} = (M, \mathcal{U})$. For clarity, we state that definition as applied to the mixed extension here.

Definition 12 *A game $\widetilde{G} \in \widetilde{\mathcal{G}}$ satisfies* superior payoff matching (SPM) *if for each player $i \in N$ and all $\mu \in M$ there exists a sequence $\{\mu_i^k\} \subset M_i$ such that* $\liminf_k U_i(\mu_i^k, \mu_{-i}) \geq \overline{\Pi}_i(\mu)$.

As a consequence of the expected utility, we could equivalently state that a game $\widetilde{G} \in \widetilde{\mathcal{G}}$ satisfies SPM if for each player $i \in N$ and all $\mu \in M$ there exists a sequence $\{x_i^k\} \subset X_i$ such that $\liminf_k \int u_i(x_i^k, x_{-i}) d\mu_{-i} \geq \overline{\Pi}_i(\mu)$. While the two formulations are equivalent, the latter is often more useful in application. The following theorem extends the result of Theorem 1 to mixed strategy equilibria.

Theorem 6 *Suppose that $\widetilde{G}(u)$ satisfies SPM. Then $\widetilde{EQ}(u) = \widetilde{IE}(\mathcal{U}_u)$ and $U(\mu) = V(\mu) = \overline{\Pi}(\mu)$ for all $u \in \mathcal{U}$, all $v \in \mathcal{U}_u$ and all $\mu \in \widetilde{EQ}(u)$. Consequently, if $\widetilde{G}$ satisfies SPM for all $\widetilde{G} \in \widetilde{\mathcal{G}}$ and $\Sigma(u) = \Sigma(v)$ for all $u, v \in \mathcal{U}$, then $\widetilde{\mathcal{G}}$ is an invariant class.*

The following theorem extends the statement of Theorem 2 to apply to the mixed extension. That is, if the mixed extension of a game satisfies SPM, so all the games in its class with weakly higher payoffs.

Corollary 3 (to Theorem 2) *Let $\mathcal{G}$ be a class of games. If $\widetilde{G}(u)$ satisfies SPM for some $u \in \mathcal{U}$, then $\widetilde{G}(v)$ satisfies SPM for all measurable functions v with $u \leq v \leq \overline{\pi}$. In particular, if $\widetilde{G}(\underline{\pi})$ satisfies SPM, then $\widetilde{\mathcal{G}}$ satisfies SPM.*

5.2 Existence of Mixed Strategy Equilibrium for Classes of Games

In this section we present two distinct approaches to obtain new results for existence of mixed strategy equilibrium. The first approach is to apply the results of Section 4 to the mixed extension of the game. The second approach is to apply the results of Section 3 and appeal to the endogenous sharing rule existence result of Simon and Zame (1990). The conditions of each approach are slightly different, while neither set of condition is technically stronger than the other.[18] Despite the disparate approaches of these two papers, we will show that the matching and efficiency conditions that we require to prove existence of equilibrium with either approach yield nearly identical results.

For the remainder of this section, we assume that X is a compact Hausdorff space and that the set of player is finite, where we write $N = \{1, 2, ..., n\}$. We now present the extension of Theorem 4 to mixed strategy equilibrium.

Theorem 7 *Consider a class of games $\widetilde{\mathcal{G}}$ and $G(u) \in \mathcal{G}$. If $\widetilde{G}(\underline{u})$ satisfies SPM and $\widetilde{G}(u)$ satisfies WRUSC, then there exists an invariant mixed strategy Nash equilibrium $\mu^* \in \widetilde{IE}(\mathcal{U}_u)$.*

This theorem follows as a corollary from Theorem 4 applied to the mixed extension.

In order to show existence of equilibrium for an endogenous sharing rule using S&Z, we must formally introduce their structure of the class of payoffs. This structure is subset of a

[18] By using the results of Section 4, we require the assumption of WRUSC, but only need $\widetilde{G}(\underline{u})$ to satisfy SPM. By using the results of Section 3, we need no WRUSC assumption, but need $\widetilde{G}(\overline{\pi})$ to satisfy SPM.

game's associated class, with some subtle differences. S&Z begin with a game $G = (X, u)$, then consider the universe of payoff possibilities, which consists of all measurable payoffs functions selected from the convex hull of the closure of the graph of the game at each strategy profile. That is, all measurable payoff functions v such that $(x, v(x)) \in \text{co}(\text{cl}G(x))$, where $\text{cl}G(x) = \{(x, r) \in X \times \mathbb{R}^N : (x, r) \in \text{cl}G\}$. Label this set of payoffs $\mathcal{U}^{SZ}$.[19] S&Z show that a mixed strategy equilibrium exists for some payoff function within the class of games they consider. This result is summarized by the following fact.

Fact 5 (Simon and Zame (1990)) *Let $G = (X, u)$ be a compact game. Then $\widetilde{EQ}(v) \neq \emptyset$ for some $v \in \mathcal{U}^{SZ}$.*

S&Z's result required that X be a metric space, however, this fact is only used to construct a finite approximation of the strategy space. Demonstrating that a compact Hausdorff space can be approximated with finite sets in general is a nontrivial task, however, this has been shown in a more recent paper by Kopperman and Wilson (1997). Thus, we present the result with Hausdorff spaces instead of metric spaces.

This result is very powerful, as it imposes almost no restrictions on the payoffs, however, it can be very difficult to apply since there is no way of knowing which sharing rule $v \in \mathcal{U}^{SZ}$ possesses a mixed strategy equilibrium. However, this is an ideal setting to apply the invariance results of Section 3, as those results can allow us to identify games that do possess an equilibrium. The following preliminaries are needed to identify these games.

For any payoff function $u \in \mathcal{U}$, define the set

$$\Psi(u) = \{x \in X : (x, \overline{\pi}(x)) \in \text{cl}G(u)\}$$

and the subclass

$$\mathcal{U}^{SZ}_{\Psi} = \{u \in \mathcal{U}^{SZ} : u(x) = \overline{\pi}(x) \text{ for all } x \in \Psi(u)\}.$$

[19] While it is always the case that S&Z's class of games is a subset of our associated class, the reverse is only true for the degenerative case that u is continuous on X. This more general construction of associated games is useful for two reasons. First, it allows for payoffs at discontinuities which cannot be obtained via a strict randomization of the utilities, as can occur when randomization or division of payoffs at the discontinuities does not occur directly on the utilities. For example, in models of price competition, payoffs at the discontinuity points are determined via a rationing of demand between the firms instead of a randomization between the minimal and maximal profits. If firms have nonlinear cost functions, then their profits may lie outside the convex hull of the closure of the continuities of the game (see for example Allison and Lepore (2016)). Second, the associated class allows for payoffs which may not be approached by payoffs at points of continuity. For example, in Bertrand price competition win nonlinear cost functions, there may be equilibria in which all firms set the same price which is below what their marginal cost would be if they served the entire market. By dividing the market, they each produce a low quantity and earn a positive profit. At a slightly lower price, any firm takes the whole market and a loss, while at a slightly higher price the firm earns no profit. Thus, profits at such discontinuities are discretely larger than at any point of continuity nearby (See for example Hoernig (2007)).

The set $\Psi(u)$ is the set of all strategy profiles x for which the highest payoff $\overline{\pi}_i(x)$ is simultaneously attainable for all players i in the closure of the graph of the game. In the Simon and Zame construction, this includes all points of continuity of u, but can also contain discontinuity points as well. In application, this would typically represent the set of strategy profiles for which it is feasible for all players you receive their highest possible payoff at that point. The subclass $\mathcal{U}_\Psi^{SZ}$ is simply the set of all payoff functions that simultaneously give all players the highest feasible payoff $\overline{\pi}_i$ whenever possible. As it turns out, $\mathcal{U}_\Psi^{SZ}$ is the class of payoff functions for which we are able to guarantee existence of equilibrium using S&Z's result.

The following theorem uses S&Z's result combined with Corollary 1 to obtain existence of an invariant equilibrium.

Theorem 8 *Let $G = (X, u)$ be a compact game and consider the class of games $\mathcal{G} = (X, \mathcal{U}^{SZ})$. If $\widetilde{G}(v)$ satisfies SPM for all $v \in \mathcal{U}^{SZ}$, then $\widetilde{IE}(\mathcal{U}_\Psi^{SZ}) \neq \emptyset$. If in addition $\Sigma(v) = \Sigma(w)$ for all $v, w \in \mathcal{U}^{SZ}$, then $\widetilde{IE}(\mathcal{U}^{SZ}) \neq \emptyset$.*

Proof. This follows immediately from Fact 5 and Corollary 6. ■

The restriction here to the class $\mathcal{U}_\Psi^{SZ}$ is because $\Sigma(u) \subset \Sigma(v)$ for all $u \in \mathcal{U}_\Psi^{SZ}$ and $v \in \mathcal{U}^{SZ}$. Thus, regardless of which rule has an equilibrium, our invariance results imply that this equilibrium is shared by the games in $\mathcal{U}_\Psi^{SZ}$.

Reny and S&Z take two very different approaches to obtaining existence of equilibrium. Despite this, the results that we have presented using the two approaches look remarkably similar. This is particularly interesting since the two approaches of these papers are so distinct. Reny's result presents conditions under which a particular game has an equilibrium. S&Z's result demonstrates that an equilibrium can always be found for some perturbation of any game. When we assume that SPM is satisfied by all games within an associated class, these two existence results are nearly identical. The only difference is that using Reny's result requires WRUSC of the mixed extension, while using S&Z's result requires that the game satisfy a weak efficiency condition. Upon further inspection, this difference is incredibly minor, as Theorem 9 of the next section guarantees that any such game satisfies WRUSC. It is easy to construct an example for which the mixed extension is WRUSC that does not also satisfy our efficiency condition. From this perspective the existence result that uses Reny's approach is slightly more general than that using S&Z. This generality is unimportant for practical application, as verification of WRUSC in the mixed extension can be very difficult in the absence of our efficiency condition. Therefore, in classes of games satisfying SPM, Reny's conditions for existence of equilibrium may identify the particular games which may have been selected by S&Z's endogenous sharing rule equilibrium.

5.3 Additional Properties of SPM in Mixed Strategies

In this subsection, we present some analogues of results in previous sections as well as verifiable sufficient conditions for SPM and WRUSC of the mixed extension. Specifically, we show that the property of SPM can be used to show that the mixed extension satisfies WRUSC, and as a special case, SPM links RUSC of a game to WRUSC of its mixed extension. Finally, to ease verification of all these conditions, we present a sufficient condition for SPM of the mixed extension that can be verified entirely in the space of pure strategies of the game. Together, these results allow for straightforward verification of SPM, WRUSC, and existence of equilibrium in the mixed extensions of games.

We begin with the relation of SPM and WRUSC. As in the previous section, let $\Psi(u)$ denote the set of strategy profiles for which maximal payoffs are feasible,

$$\Psi(u) = \{x \in X : (x, \overline{\pi}(x)) \in \mathrm{cl}G(u)\}.$$

Theorem 9 *Consider a game $G(u)$ and its mixed extension $\widetilde{G}(u)$. If $\widetilde{G}(u)$ satisfies SPM and $u(x) = \overline{\pi}(x)$ for all $x \in \Psi(u)$, then $\widetilde{G}(u)$ satisfies WRUSC.*

Proof. Let $(\mu^*, u^*) \in \mathrm{cl}\widetilde{G}(u) \diagdown \widetilde{G}(u)$. First, since $u(x) = \overline{\pi}(x)$ for all $x \in \Psi(u)$, it follows that $\Sigma(u) \subset X \diagdown \Psi(u)$. If $\mu(\Sigma(u)) = 0$, then since $(\mu^*, u^*) \notin \widetilde{G}(u)$, it follows that $u_i^* < \int \overline{\pi}_i(x) d\mu^*$ for some player i. Thus, there is some $x_i \in X_i$ such that $\int \overline{\pi}_i(x_i, x_{-i}) d\mu_{-i}^* > u_i^*$. From Theorem 10, player i can match a payoff of $\int \overline{\pi}_i(x_i, x_{-i}) d\mu_{-i}^*$ at μ^*, and so player i has a strategy $x_i' \in X_i$ such that $u_i(x_i, \mu_{-i}^*) > u_i^*$.

Suppose that $\mu(\Sigma(u)) > 0$. Let $A : X \to \mathbb{R}$ be the aggregation of the payoffs of each player, that is, $A(x) = \sum u_i(x)$. Then define the upper semicontinuous envelope of A, $\overline{A}$ in the same way as $\overline{u}$, noting that $\overline{A}(x) \leq \sum \overline{\pi}(x)$ and that $\overline{A}$ is upper semicontinuous. Thus, $\int \overline{A}(u) d\mu$ is upper semicontinuous in μ. It follows that $\sum u_i^* \leq \int \overline{A}(x) d\mu^*$. We will show that $\int \overline{A}(x) d\mu^* < \int \sum \overline{\pi}(x) d\mu$. This will imply that $u_i^* < \int \overline{\pi}_i(x) d\mu^*$ for some player i, and the argument above will then guarantee that $\widetilde{G}(u)$ satisfies WRUSC. Note that $\overline{A}(x) = \sum \overline{\pi}(x)$ if and only if $x \in \Psi(u)$. Thus, since $\Sigma(u) \cap \Psi(u) = \emptyset$, it follows that $\overline{A}(x) < \sum \overline{\pi}(x)$ for all $x \in \Sigma(u)$. It follows immediately that $\int \overline{A}(x) d\mu^* < \int \sum \overline{\pi}(x) d\mu$. We conclude that $\widetilde{G}(u)$ satisfies WRUSC.

Next, suppose that $\widetilde{G}(u)$ satisfies WRUSC. Suppose that $u(x) \neq \overline{\pi}(x)$ for some $x \in X$. Then consider the strategy profile μ defined by $\mu(\{x\}) = 1$. Then note that $(\mu, \overline{\pi}(x)) \in \mathrm{cl}\widetilde{G}(u)$ and. ■

Remark 6 *This result provides a method for verifying WRUSC in the mixed extension of games. As a special case, note that if a game $G(u)$ satisfies reciprocal upper semicontinuity (RUSC), then $\Sigma(u) \cap \Psi(u) = \emptyset$. Thus, SPM provides a connection between RUSC of a game and WRUSC of it's mixed extension.*

While this result can reduce the burden of verifying WRUSC, it still requires that one verify SPM in the mixed extension, which can still be challenging. We alleviate this burden by presenting an easily verifiable sufficient conditions for SPM. This condition, uniform superior payoff matching, is advantageous in that it only requires one to check the payoffs at pure strategy profiles.

Definition 13 *A game $G \in \mathcal{G}$ satisfies* uniform superior payoff matching (USPM) *if for each player $i \in N$ and all $x_i \in X_i$, there exists a sequence $\{x_i^k\} \subset X_i$ such that* $\liminf_k u_i(x_i^k, x_{-i}) \geq \overline{\pi}_i(x_i, x_{-i})$ *for all $x_{-i} \in X_{-i}$.*

A game satisfies USPM if each player can match their highest possible payoff against *any* strategy profile for the other players with a single sequence of deviations. USPM is typically quite intuitive and easy to verify in games that satisfy it. A consequence of USPM being a stronger condition than SPM is that it guarantees a stricter payoff characterization. The following theorem summarizes these results.

Theorem 10 *Suppose that $G(u) \in \mathcal{G}$ satisfies USPM. Then at any $\mu \in M$, each player i can match a payoff of $\int \overline{\pi}_i(x)d\mu$ at μ. Consequently, in any equilibrium $\mu \in I\widetilde{E}(u)$, it must be that $U(\mu) = \int \overline{\pi}(x)d\mu$.*

Proof. Let $\mu \in M$ and $x_i \in X_i$. Since $\mathcal{U}$ is uniformly bounded, we may assume without loss of generality that $\underline{\pi}(x) \geq 0$ for all $x \in X$. From USPM, let x_i^k be such that $\liminf_k u_i(x_i^k, x_{-i}) \geq \overline{\pi}_i(x_i, x_{-i})$ for all $x_{-i} \in X_{-i}$. Then note that $f_i^k(x_{-i}) \equiv u_i(x_i^k, x_{-i})$ is a sequence of nonnegative functions. Fatou's Lemma thus implies that

$$\liminf_k \int f_i^k(x_{-i})d\mu_{-i} \geq \int \liminf_k f_i^k(x_{-i})d\mu.$$

Since $\liminf_k \int f_i^k(x_{-i})d\mu_{-i} = \liminf_k \int u_i(x_i^k, x_{-i})d\mu_{-i}$ and $\liminf_k f_i^k(x_{-i}) \geq \overline{\pi}_i(x_i, x_{-i})$ for all $x_{-i} \in X_{-i}$, then we conclude that

$$\begin{aligned} \liminf_k \int u_i(x_i^k, x_{-i})d\mu_{-i} &\geq \int \liminf_k u_i(x_i^k, x_{-i})d\mu \\ &\geq \int \overline{\pi}_i(x_i, x_{-i})d\mu. \end{aligned}$$

Therefore, player i can match a payoff of $\int \overline{\pi}_i(x)d\mu$ at μ. The equilibrium payoff result follows immediately. Finally, the upper semicontinuity of $\overline{\pi}$ guarantees that $\overline{U}(\mu) \leq \int \overline{\pi}(x)d\mu$, further implying that $\widetilde{G}(u)$ satisfies SPM. ■

Remark 7 *This result is particularly powerful when combined with the other results of this section. Note that one only needs to check that a game $G(\underline{u})$ satisfies USPM and that*

$u(x) = \overline{\pi}(x)$ for all $x \in \Psi(u)$, which allows for a more intuitive understanding of whether the conditions are satisfied. With Theorems 7, 8, 9, and 10 it is possible to verify that the mixed extension satisfies SPM, WRUSC, and possesses a Nash equilibrium without any computations in the mixed extension.

While SPM is often more difficult to verify, it is satisfied by more games than USPM. This is formally demonstrate by Example 7, located in the appendix.

The following corollary extends the statement of Theorem 2 to apply both to the mixed extension as well as to USPM. That is, if a game satisfies USPM, so will all games with weakly higher payoffs.

Corollary 4 (to Theorem 2) *Let $\mathcal{G}$ be a class of games. If $G(u)$ ($\widetilde{G}(u)$) satisfies USPM (SPM) for some $u \in \mathcal{U}$, then $G(v)$ ($\widetilde{G}(v)$) satisfies USPM (SPM) for all measurable functions v with $u \leq v \leq \overline{\pi}$. In particular, if $G(\underline{\pi})$ ($\widetilde{G}(\underline{\pi})$) satisfies USPM (SPM), then $\mathcal{G}$ ($\widetilde{\mathcal{G}}$) satisfies USPM (SPM).*

For games in pure strategies, the results of Section 3 imply that if $x \in X$ is an equilibrium, then $x \notin \Sigma(u)$. While technically meaningful, this fact is of little use in such a setting. When applied to the mixed extension of games, the statement $\mu \notin \widetilde{\Sigma}(u)$ has a much stronger and more important implication as related to the pure strategies of the game. This relationship is demonstrated in the following theorem.

Theorem 11 *Suppose that $\overline{\pi}$ is measurable and that $\overline{\Pi}(\mu) = \int \overline{\pi}(x)d\mu$ for all $\mu \in M$. If the mixed extension $\widetilde{G}(u) \in \widetilde{\mathcal{G}}$ satisfies SPM, then $\mu(\Sigma(u)) = 0$ for all $\mu \in \widetilde{EQ}(u)$.*

Proof of Theorem 11. Let $\mu \in \widetilde{EQ}(u)$. Then from Corollary 6, $U(\mu) = \overline{\Pi}(\mu)$. Thus, our assumption guarantees that $U(\mu) = \int \overline{\pi}(x)d\mu$, or rather, $\int u(x)d\mu = \int \overline{\pi}(x)d\mu$. It follows immediately that $\mu(\Sigma(u)) = 0$. ■

Thus, a critical property that is implied by equilibrium invariance is that $\mu(\Sigma(u)) = 0$ for all $\mu \in \widetilde{EQ}(u)$. For application to discontinuous games, this implies that discontinuities at which some player does not receive her highest possible payoff occur with probability zero in any invariant equilibrium. While it may seem as though $\mu(\Sigma(u)) = 0$ is sufficient to imply invariance, the following example demonstrates that this is not the case.

Example 5 *Consider a two player all-pay auction with budget constraints. Player 1 has a budget constraint of 1 and player 2 has a budget constraint of 2. Thus, each player i simultaneously selects bids in $X_i = [0, i]$. A prize of common value 2 is awarded to the player* *with the highest bid, with ties being broken by some tie breaking rule. Each player must pay her bid regardless of whether she wins the prize. The payoffs can thus be expressed as*

$$u_i(x) = 2P_i(x) - x_i,$$

where $P_i(x)$ is the probability that player i wins, defined by

$$P_i(x) = \begin{cases} 1 & \text{if} \quad x_i > x_j \\ \alpha_i(x) & \text{if} \quad x_i = x_j \\ 0 & \text{if} \quad x_i < x_j \end{cases},$$

where $\alpha_i : X \to [0, 1]$ is measurable and $\alpha_1(x) + \alpha_2(x) = 1$. Note that all possible tie breaking rules α naturally define a class of auctions. We will consider that class. Note that $\Sigma(u) = \{x \in X : x_1 = x_2\}$ for any payoff function u in this class.

In this class of auctions, any equilibrium must have $\mu(\Sigma(u)) = 0$ and $\mu_2(\{1\}) = 1/2$, regardless of the sharing rule. Given any such strategy profile, player 1 has an expected payoff of zero. However, if $\alpha_1(1, 1)$ is sufficiently large, player 1 has a profitable deviation from any such strategy profile to $x'_1 = 1$, as this earns a positive expected payoff. Therefore, the equilibrium is not invariant despite the fact that $\mu(\Sigma(u)) = 0$.

Note that the property that prevents invariance in this example, despite the set of discontinuities being measure zero in equilibrium, is that some player i possesses a pure strategy x_i such that the other player's mixed strategy assigned a positive probability to the set of strategies at which $u_i(x_i, x_{-i})$ is discontinuous. While the equilibrium assigns zero mass to this event, this is because player i chooses x_i with probability zero. In this case, there exists an alternative payoff function for which player i has a profitable deviation to a strategy that was not in the support of her equilibrium strategy under the initial payoff function.

6 Applications

In this section we define a game that accommodates as special cases many of the applications found in the literature on discontinuous games and then use this game to show the intuitive restrictions imposed by our conditions. The game we consider is one in which there are L types of discontinuities that determine each players payoffs.[20] Each type of discontinuity can be interpreted in many ways depending on the applications. For example, these induced by L simultaneous auctions or contests (as is the case with a multiple battlefields Colonel Blotto game: Roberson (2006), and Roberson and Kvasov (2011)), or they could be discontinuities in demand induced by the indifference of L different consumers in a multi-consumer version of the Catalog competition game (Page and Monteiro (2003), and Monteiro and Page (2007,

[20] The approach used for the examples below would work for any countable number of discontinuity types.

2008)).[21] While we keep the structure of this game intentionally general, we use specific examples to clearly demonstrate how our matching conditions and results can be applied.

Let X be a compact, convex Hausdorff space endowed with a partial order $\succsim$. With some abuse of notation, we will also use $\succsim$ to represent the partial order on X_i. For each player i, there are L bounded and continuous functions $f_i^l : X \to \mathbb{R}$, $l \in \{1, 2, ..., L\}$ that identify the points of discontinuity. Specifically, if $u_i(\cdot)$ is discontinuous at x, then $f_i^l(x) = 0$ for some l. Note that this is a necessary condition only, it may be the case that $f_i^l(x) = 0$ at a point of continuity x. Each player i has a payoff function in a class defined as follows. Define the vector $\Lambda^x = (\Lambda_1^x, ..., \Lambda_L^x)$ such that for each l, $\Lambda_l^x = 1$ if $f_i^l(x) \geq 0$ and $\Lambda_l^x = 0$ otherwise. Similarly, define $\Lambda_x = (\Lambda_{x,1}, ..., \Lambda_{x,L})$ such that for each l, $\Lambda_{x,l} = 1$ if $f_i^l(x) > 0$ and $\Lambda_{x,l} = 0$ otherwise.[22] We use the notation $\Lambda^x(j)$ and $\Lambda_x(j)$ anytime referring these sets for a specific player j other than the arbitrary player i. Note that $\Lambda^x = \Lambda_x$ for any x without the possibility of a discontinuity and that $\Lambda^x > \Lambda_x$ for any x with the possibility of discontinuity.[23] At any point $x \in X$ such that $f_i^l(x) \neq 0$ for all l, player i's payoff is $u_i(x) = \varphi_i(x, \Lambda^x)$. For any $x \in X$ such that $f_i^l(x) = 0$ for some l, $u_i(x) = \theta_i(x, \Lambda^x)$. The set of sharing rules we consider only differ based on the specification of $\theta_i(x, \Lambda^x)$. The following assumptions on φ and θ define the basic properties of the set of payoff functions $\mathcal{U}$.

A.1 For each player i and all $\Lambda \in \{0, 1\}^L$, $\varphi_i(x, \Lambda)$ is uniformly continuous on X.

A.2 For each player i and all $\Lambda' \geq \Lambda$, $\varphi_i(x, \Lambda') \geq \varphi_i(x, \Lambda)$ for all $x \in X$.

A.3 For each player i and all $x \in X$, $\varphi_i(x, \Lambda^x) \geq \theta_i(x, \Lambda^x)$.

Based on $A.1 - A.3$, we may compute upper and lower semicontinuous envelopes of the payoffs as

$$\begin{aligned} \overline{\varphi}_i(x) &= \varphi_i(x, \Lambda^x), \\ \underline{\varphi}_i(x) &= \varphi_i(x, \Lambda_x). \end{aligned}$$

The next assumption provides sufficient restriction to show this class of games satisfies SPM.

[21] Although we use the contest and oligopoly examples for interpretation, it is worth noting that much of the literature on economic applications of discontinuous games nests within the framework of our example. This includes for example the following recent applications: *All-pay contests*: Siegel (2009, 2011); *Bertrand-Edgeworth price competition*: Allison and Lepore (2016); *Price competition with limited comparability:* Piccione and Spiegler (2012); ; *Spatial voting*: Duggan (2007); *Network attack and defense*: Kovenock and Roberson (2017); *Peer Pressure*: Calvó-Armengol and Jackson (2010); *Pre-marital investment*: Peters (2007) ; *Labor market search*: Galenianos and Kircher (2012); *Competitive matching*: Damiano and Li (2008).

[22] Note that Λ^x and Λ_x as defined depend on the player i. We suppress the notation for this as there is no need to consider multiple players' sets simultaneously, and thus there is no ambiguity.

[23] Here we use the notation $\Lambda^x > \Lambda_x$ to denote that $\Lambda^x \geq \Lambda_x$ and $\Lambda_l^x > \Lambda_{x,l}$ for at least one l.

A.4 For any $x \in X$, either

(i) there is an $x_i' \in X_i$ in every neighborhood of x_i such that $f_i^l(x_i', x_{-i}) > f_i^l(x_i, x_{-i})$ for all l such that $f_i^l(x) = 0$, or

(ii) there is an $x_i' \in X_i$ such that $\underline{\varphi}_i(x_i', x_{-i}) \geq \overline{\varphi}_i(x)$.

Invariance

Proposition 1 *Suppose that A.1-A.4 are satisfied. Then the application game $(X, \underline{u})$ satisfies SPM.*

Proof. From A.1-A.3, $\overline{u}_i(x) \leq \overline{\varphi}_i(x)$ and $\underline{u}_i(x) \geq \underline{\varphi}_i(x)$ for all $u \in \mathcal{U}$. This also implies that $\overline{\pi}_i(x) \leq \overline{\varphi}_i(x)$. If A.4(i) holds at a point x, then for all i, l, and $x \in X$ there is a sequence $x_i^k \to x_i$ such that $f_i^l(x_i^k, x_{-i}) > 0$ for all k for all l with $f_i^l(x) = 0$. By the continuity of each f_i^l, it follows that for sufficiently large k, $f_i^l(x_i^k, x_{-i}) < 0$ for all l such that $f_i^l(x) < 0$ and $f_i^l(x_i^k, x_{-i}) > 0$ for all l such that $f_i^l(x) > 0$. Therefore, for sufficiently large k, $u_i(x_i^k, x_{-i}) = \varphi_i(x_i^k, x_{-i}, \Lambda^{x_i^k, x_{-i}})$, and thus $\lim_k u_i(x_i^k, x_{-i}) = \overline{\varphi}_i(x)$. Finally, since $f_i^l(x_i^k, x_{-i}) \neq 0$ for all k sufficiently large, it follows that $\Lambda^{x_i^k, x_{-i}} = \Lambda_{x_i^k, x_{-i}}$, so $u_i(x_i^k, x_{-i}) = \underline{\varphi}_i(x_i^k, x_{-i})$ for sufficiently large k. Thus, for all x_i satisfying (i):

$$\begin{aligned}
\lim_k \underline{u}_i(x_i^k, x_{-i}) &\geq \lim_k \underline{\varphi}_i(x_i^k, x_{-i}) \\
&= \lim_k \varphi_i(x_i^k, x_{-i}, \Lambda^{x_i^k, x_{-i}}) \\
&= \overline{\varphi}_i(x) \\
&\geq \overline{\pi}_i(x).
\end{aligned}$$

Therefore, the conditions of SPM are satisfied at x.

For use in application, A.4(ii) is intended to address boundary points such that every neighborhood around a point x_i is not fully contained in X_i. At such a point this condition guarantees a way to match the upper bound payoff that may not correspond to a local deviation. Note that A.4 (ii) directly guarantees SPM for all such boundary points.

We conclude that if A.4 is satisfied, then the game $(X, \underline{u})$ satisfies SPM. ∎

Assumption A.4 highlights a fundamental property that is required for the verification of SPM: that all discontinuity types be avoided via deviations in the same direction. In this general setting, this corresponds to the existence of local deviations which increase f_i^l for each relevant discontinuity type l. It is precisely under this condition that $\overline{u}_i(x) = \overline{\varphi}_i(x)$. Otherwise, it may be the case that $\overline{u}_i(x) < \overline{\varphi}_i(x)$, and this may lead to nonexistence of equilibrium, as is the case in the example presented Sion and Wolfe (1957). They present game in which each player chooses a strategy in an interval, and the “good” side of discontinuities may correspond to either an increase or a decrease in choice depending on the other player's strategy. Consequently, neither A.4 nor SPM can be satisfied.

Next we consider conditions such that the application game satisfy USPM. Assumptions A.5 and A.6 are stronger than A.4 as they must guarantee the same sequence can match the upper bound payoff for all pure strategies of the other players.

A.5 For all x and l such that $f_i^l(x) = 0$, if $x_i' \succ x_i$, then $f_i^l(x_i', x_{-i}) > f_i^l(x_i, x_{-i})$. For all x and l such that $f_i^l(x) \neq 0$, if $x_i' \succ x_i$, then $f_i^l(x_i', x_{-i}) \geq f_i^l(x_i, x_{-i})$.

Now we must consider boundary points $x \in X$ such that there is no $x_i' \in X_i$ such that $x_i' \succ x$. Define the set $\overline{B}(X_i) = \{x_i \in X_i : x_i' \not\succ x_i \text{ for all } x_i \in X_i\}$. We make the assumption that there is some way to secure at least the payoff at $\overline{\varphi}_i(x)$ at any boundary point x. While this is stronger than necessary, we make this assumption for clarity of exposition. This choice is justified by the fact that this assumption is satisfied in the intended applications.

A.6 For all i, any $x_i \in \overline{B}(X_i)$, there is an $x_i' \in X_i$ such that for all l, $\underline{\varphi}_i(x_i', x_{-i}) \geq \overline{\varphi}_i(x_i, x_{-i})$ for all $x_{-i} \in X_{-i}$.

Proposition 2 *Suppose that A.1-A.3, A.5 & A.6 are satisfied. Then the game $(X, \underline{u})$ satisfies USPM.*

Proof. A.5 guarantees that for all i and $x \in X$ such that there exists an $x_i' \in X_i$ and $x_i' \succ x_i$, $\overline{u}_i(x) = \overline{\varphi}_i(x)$. Thus, $\overline{\pi}_i(x) = \overline{\varphi}_i(x)$. Further, it guarantees that for all i and $x_i \in X_i$ with $x_i' \in X_i$ such that $x_i' \succ x_i$ there is a sequence $x_i^k \succ x_i$ with $\Lambda^{x_i^k, x_{-i}} = \Lambda_{x_i^k, x_{-i}} = \Lambda^{x_i, x_{-i}}$ and $f_i^l(x_i^k, x_{-i}) \neq 0$ for all l and all $x_{-i} \in X_{-i}$. It follows that

$$\begin{aligned} \lim_k \underline{u}_i(x_i^k, x_{-i}) &= \overline{\varphi}_i(x) \\ &\geq \overline{\pi}_i(x) \end{aligned}$$

for all $x_{-i} \in X_{-i}$. Thus, the conditions of USPM are satisfied for all $x_i \notin \overline{B}(X_i)$.

A.6 directly gives us USPM for all $x_i \in \overline{B}(X_i)$. Therefore, this game satisfies USPM. ■

Existence

Now define the class of payoff functions $\mathcal{U}^{SZ}$ as the closure of graph of $(X, u(x))$, where $u(x) = (u_1(x), ..., u_N(x))$. Based on the structure of the game, the closure of the graph of each game $G(v)$ is the same for all sharing rules $v \in \mathcal{U}^{SZ}$ and $\overline{u}(x) = \overline{\pi}(x)$. Denote by $\mathcal{U}_\Psi^{SZ}$ the class of all payoff selections $v \in \mathcal{U}^{SZ}$ in the closure of the graph of the game such that $v(x) = \overline{u}(x)$, if $(x, \overline{u}(x))$ is in the closure of the graph of the game. This construction is sufficient to guarantee $G(v)$ is RUSC for all $v \in \mathcal{U}_\Psi^{SZ}$. Denote by $\mathcal{G}_\Psi^{SZ}$ the class of games with payoffs $v \in \mathcal{U}_\Psi^{SZ}$. This leads to the following results on existence of invariant equilibrium.

Proposition 3 *Suppose that A.1-A.4 are satisfied and that there is a* $u \in \mathcal{U}_{\Psi}^{SZ}$ *that is quasi-concave. Then* $IE(\mathcal{U}_{\Psi}^{SZ}) = EQ(\mathcal{U}_{\Psi}^{SZ}) \neq \emptyset$.

Similarly we have the following result for existence of invariant mixed strategy equilibrium.

Proposition 4 *Suppose that A1-A3, A5 & A6 are satisfied. Then* $\widetilde{IE}(\mathcal{U}_{\Psi}^{SZ}) = \widetilde{EQ}(\mathcal{U}_{\Psi}^{SZ}) \neq \emptyset$.

This application game is still fairly abstract, so we now provide two examples of games that fall within this framework. Before turning to the examples the following remark addresses a game structure that falls within our general assumptions, but is not included in the application game.

Remark 8 *The assumption that payoffs are uniformly continuous on the set of continuity points inherently rules out certain types of games. One common example is a simple Tullock lottery contest. In such a contest, players bid for a chance to win a prize, where the probability of winning is equal to one's share of the total bids. An issue arises when no one bids. In such a setting, there is one discontinuity type identified by* $f(x) = 0$ *where* $f(x) = \sum_i x_i$. *However, the payoffs when* $f(x) > 0$ *do not correspond to a uniformly continuous function* $\varphi : X \to \mathbb{R}$. *To see why, consider sequences* $x^k(i)$ *defined by* $x_i^k(i) = 1/k$ *and* $x_{-i}^k(i) = (0, ..., 0)$. *Note that* $\lim_k u_i(x^k(i)) = V$, *where* V *is the value of the prize, while* $\lim_k u_i(x^k(j)) = 0$ *for* $j \neq i$. *Even though this game and games with this property fail to satisfy the structure of this section, they can still satisfy the conditions of this paper. Indeed, for this example, the sequence* $x_i^k(i)$ *above can be used to show that this game satisfies SPM.*

6.1 Rank Order Contest

Consider a game in which all player compete in a simultaneous contest with each player's payoffs determined by their ranking as determined by their actions relative to those of the other players. Additionally, we allow a player's payoff to depend on which players she outperforms, as well as on the actions of all the players.

Formally, each player picks an $x_i \in X_i$, a nonempty, convex and compact subset of $\mathbb{R}_+^n$. The performance of player i relative to player j is evaluated by $f_i^j(x)$, with $f_i^j(x) = 0$ if the performance of players is equivalent and $f_i^j(x) > 0$ if i outperforms j. Each player i receives a prize $m_i(x, \Lambda)$, where $\Lambda^x \geq \Lambda \geq \Lambda_x$. In this context, the set Λ^x corresponds to the set of players that a player performs at least as well as, while the set Λ_x corresponds to the set of players that she outperforms. In the event that i performs equivalently to some other player j ($f_i^j(x) = 0$), the rank is assigned randomly, hence the necessity of specifying $\Lambda^x \geq \Lambda \geq \Lambda_x$. Each player i pays effort cost $c_i(x_i) \geq 0$, which is nondecreasing in x_i. We assume that m_i and c_i are continuous (and thus uniformly continuous) on X.

R.1 Players always prefer a higher rank: for any $x \in X$ and $\Lambda' \geq \Lambda$, $m_i(x, \Lambda') \geq m_i(x, \Lambda) \geq 0$.[24]

As constructed, the set of possible the utility functions of player i at all points of continuity x is given by $\varphi_i(x, \Lambda^x) = m_i(x, \Lambda^x) - c_i(x)$ and at points of discontinuity by $\theta_i(x, \Lambda^x) = \sum_{\Lambda \in [\Lambda_x, \Lambda^x]} \alpha_i(x, \Lambda)\varphi_i(x, \Lambda)$ for some set of values $\alpha_i(x, \Lambda) \in [0, 1]$ with $\sum_{\Lambda \in [\Lambda_x, \Lambda^x]} \alpha_i(x, \Lambda) = 1$. Thus, the payoffs of this game satisfy the basic structure of the application game.

R.2 Efforts are strategically bounded: for each player i, there exists a $z \in X_i$ with $\{x_i \in X_i : x_i > z\} \neq \emptyset$ such that for all $x_{-i} \in X_{-i}$ and $\Lambda \in \{0, 1\}^L$, $m_i(x, \Lambda) - c_i(x_i) \leq 0$ if $x_i > z$.

The next assumption guarantees that A.4 is satisfied and thus SPM is satisfied.

R.3 Extra effort can break ties in performance: for all players i and j and all $x \in X$ such that $f_i^j(x) = 0$, $f_i^j(x)$ is strictly increasing in x_i at x.

The following assumption is necessary to guarantee that payoffs are quasiconcave.

R.Q For each player i, the cost of effort $c_i(x_i) = 0$ for all x_i and that each $m_i(x, \Lambda^x)$ is quasi-concave in x_i.

Proposition 5 *Suppose that R.1-R.3 and R.Q are satisfied. Then the rank order contest has an invariant equilibrium,* $IE(\mathcal{U}_\Psi^{SZ}) = EQ(\mathcal{U}_\Psi^{SZ}) \neq \emptyset$.

Proof. We show that R.1-R.2 imply A.1-A.3 and then the remainder of the proof follows directly from Proposition 1. First A.1 is just based on the uniform continuity of m_i and c_i and clearly R.1 directly implies A.2. For all $\alpha_i(x, \Lambda) \in [0, 1]$ with $\sum_{\Lambda \in [\Lambda_x, \Lambda^x]} \alpha_i(x, \Lambda) = 1$, the weighted sum $\theta_i(x, \Lambda^x) = \sum_{\Lambda \in [\Lambda_x, \Lambda^x]} \alpha_i(x, \Lambda)\varphi_i(x, \Lambda)$ is an average of payoffs $\varphi_i(x, \Lambda) \leq \varphi_i(x, \Lambda^x)$, which implies $\theta_i(x, \Lambda^x) \leq \varphi_i(x, \Lambda^x)$. From R.2 and R.3, for any x such that $\varphi_i(x, \Lambda^x) > \theta_i(x, \Lambda^x)$ player i can pick $x_i' > x_i$ and get $\varphi_i(x_i', x_{-i}, \Lambda^x)$. Based on the continuity of $\varphi_i(x, \Lambda^x)$ for $x_i' > x_i$ close enough to x_i $\varphi_i(x_i', x_{-i}, \Lambda^x) > \theta_i(x, \Lambda^x)$. At any $x_i \in \partial X_i$ with no $x_i' \in X_i$ such that $x_i' > x_i$, by R.2 $\varphi_i(x, \Lambda^x) \leq 0$. Based on the fact that $m_i(x, \Lambda) \geq 0$ and $c_i(x_i) = 0$ for all x_i, $\varphi_i(x, \Lambda^x) = 0$ and $\underline{\varphi}_i(x) = 0$. Finally, the quasiconcavity requirement follows directly from R.Q. ■

The two assumptions that follow are reasonable restrictions for the Rank Order Contest that imply A.5 and A.6.

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[24] The assumption that payoffs are nonnegative give us a boundry condition, but is not strictly necessary.

R.4 Increased effort always increases performance: For all $x \in X$ and all $x_i' \geq x_i$ with $x_i \neq x_i'$, $f_i^j(x_i', x_{-i}) > f_i^j(x_i, x_{-i})$ for all $j \neq i$.

R.5 For every player i, there is a $x_i \in X_i$ such that $c_i(x_i) = 0$.

Proposition 6 *Suppose that R.1, R.2, R.4 and R.5 are satisfied. Then the rank order contest has an invariant mixed strategy equilibrium,* $\widetilde{IE}(\mathcal{U}_\Psi^{SZ}) = \widetilde{EQ}(\mathcal{U}_\Psi^{SZ}) \neq \emptyset$.

Proof. We only need to show that R.1, R.2 and R.4 imply A.5 and A.6 to show that the conditions for Proposition 2 are satisfied. R.4 directly implies A.5. At any $x_i \in \partial X_i$ with no $x_i' \in X_i$ such that $x_i' > x_i$, by R.2 $\varphi_i(x, \Lambda^x) \leq 0$. Based on R.5 there is a $x_i' \in X$ such that $\varphi_i(x_i', x, (0, ..., 0)) \geq 0$. This means that $\underline{\varphi}_i(x) = \varphi_i(x_i', x, \Lambda_x) \geq \varphi_i(x_i', x, \{0, ..., 0\}) \geq 0$ and A.6 is satisfied. ■

6.2 Oligopoly with endogenous product lines, quality and prices

Consider a model of oligopoly with endogenous product choice, quality, and price. The framework is general enough to include quantity as a choice variable or allow it to be determined completely by consumer decisions *a la* Bertrand.

$n \geq 2$ firms simultaneously compete by choosing a product line $q_i \in [0, 1]^T$ that consists of a quality choice for up to T different products and price vector $p_i \in \mathbb{R}_+^T$ of corresponding prices for those products. We interpret a quality choice $q_i^t = 0$ for a product t as not offering that product. Additionally, the firm sets a price vector $p_i \in \mathbb{R}_+^T$. Thus, a strategy for each firm i is a vector $x_i = (q_i, p_i)$ that consists of a vector of qualities for up to T products and corresponding prices.

Market demand consists of H groups of consumers. Each group may be qualitatively different, may consist of a single consumer, or may represent a continuum of consumers that are individually small relative to the market. The demand for individual products is derived from consumer preferences. As such, there will be discontinuities in the demand facing each firm when the prices and qualities of goods are such that some consumers are indifferent between the combinations offered by multiple firms. These points of indifference are captured by continuous functions $f_i^{h^{t,j}}(x)$ that have the property that $f_i^{h^{t,j}}(x) = 0$ if the consumers in group h are indifferent between (q_i^t, p_i^t) and (q_j^t, p_j^t), with $f_i^{h^{t,j}}(x) > 0$ indicating that group h prefers (q_i^t, p_i^t) to (q_j^t, p_j^t). Thus, there are $L = HT(n-1)$ possible discontinuity types for each firm i. That is, a possible discontinuity type for each group of consumers, product, and rival firm. As in the general application framework we define the sets Λ^x and Λ_x as binary valued vectors of length L based on the functions $f_i^l(x)$.

The interpretation of each function $f_i^{h^{t,j}}(x)$ is critical to this application. Let us introduce some notation to provide a proper interpretation of each $f_i^{h^{t,j}}(x)$. Denote by $\eta_h(x_i^t; x)$ the real valued utility of consumer type h buying good type t from firm i before buying from type t from firm j and buying their optimal corresponding bundle from all firms. Using this utility we write $f_i^{h^{t,j}}(x) = \eta_h(x_i^t; x) - \eta_h(x_j^t; x)$. Thus, $f_i^{h^{t,j}}(x) = 0$ when buying good t from firm i before firm j is equivalent to buying good t from firm j before firm i.[25]

Any payoff functions is the defined by $u_i(x) = \varphi_i(x, \Lambda^x)$ for all x such that $f_i^l(x) \neq 0$ for all l, and $u_i(x) = \theta_i(x, \Lambda^x)$ if $f_i^l(x) = 0$ for at least one l. At any point of continuity, $\varphi_i(x, \Lambda_x) = \varphi_i(x, \Lambda^x)$ and there is no ambiguity as to the specification of the profit.

O.1 Firms have a greater potential for profit when facing a greater demand. Formally, for all $x \in X$, for any $\Lambda' \geq \Lambda$, $\varphi_i(x, \Lambda') \geq \varphi_i(x, \Lambda)$.

This level of abstraction is convenient because of its versatility in handling quantity of production and sale as well as any form of demand rationing. If we were to include quantity as a choice variable simultaneously with the prices and qualities, then O.1 is trivially satisfied. In a Bertrand setting where firms must serve all demand they face, this assumption is satisfied as long as marginal cost of production is constant. Alternatively, if production is determined after realization of prices, then this assumption simply requires that firms do not reduce their quantity when they face a higher demand, which is consistent with profit maximizing behavior when costs of production are weakly quasiconvex in quantities. As such, assumption O.1 will be satisfied in a large range of market structures. Interestingly, this assumption is general enough to allow demand rationing rules in which firms with more preferable quality/price bundles do not have complete priority access to consumers as long as it is still more profitable to have a more preferable quality/price bundle.

We specify the objective function of each firm i to be a profit function $u_i(x)$, allows for any demand of indifferent consumers shared in many ways at ties between firms for a consumers type.

O.2 For all $x \in X$, $\varphi_i(x, \Lambda_x) \leq \theta_i(x, \Lambda^x) \leq \varphi_i(x, \Lambda^x)$.

We need one additional assumption to manage the continuity of payoffs.

O.3 The profit of the firms is uniformly continuous in x wherever demand is continuous. Formally, $u_i(x)$ is uniformly continuous on $\{x \in X : f_i^l(x) \neq 0 \text{ for all } l = \{1, ..., L\}\}$.

This uniform continuity assumption is fairly innocuous, as in application this is implied by continuity of the cost function together with continuity of consumer preferences or market demand and rationing rule.

The next assumption serves to guarantee that as a firm attempts to implement perfect quality of any good cost becomes unprofitably large.

[25] Implicit in this construction is some rationing mechanism at each x.

O.4 For all t, $\lim_{q_i^t \uparrow 1} \varphi_i(x,(1,...,1)) < \sup_{x_i} \inf_{x_{-i}} u_i(x)$.

The next assumption is that all consumers weakly prefer lower prices and higher quality in a product and that at indifference points in which consumers buy a product t all consumers strictly prefer a lower price. To write this formally, we must introduce more notation. Consider the set of firm $J \subset N \smallsetminus i$. Define the vector $\Lambda^{x-h^{t,J}}$ such that $\Lambda^{x-h^{t,J}}_{h^{t,j}} = 0$ for all $j \in J$ and $\Lambda^{x-h^{t,J}}_l = \Lambda^x_l$ for all $l \in \{h^{t,j}\}_{j \in J}$. This is the same vector for firm i as Λ^x_l except with firm i getting last chance of all tied firm to serve the demand for good t of consumer type h.

O.5 For each h and t, $\eta_h(x_i^t; x)$ is nondecreasing in q_i^t and nonincreasing in p_i^t. For each h and t, if there is nonempty set of firms J such that $f^i_{h^{t,j}}(x) = 0$ for all $j \in J$ and $\theta_i(x, \Lambda^x) > \theta_i(x, \Lambda^{x-h^{t,J}})$, then $\eta_h(x_i^t; x)$ is strictly decreasing in p_i^t.

The conditions thus far are sufficient to guarantee that USPM is satisfied in this game. However, we still need to bound the pricing space as well as guarantee the payoff functions are RUSC. The following standard assumption bounds the pricing space.

O.6 Demand for each good vanishes at a sufficiently high, finite price. Formally, there exists a price vector $\overline{\rho}$ such that $u_i(q_i, p_i, x_{-i})$ is constant in p_i^t for all $p_i^t > \overline{\rho}^t$ and all x_{-i}, and $u_i(q_i, \overline{\rho}, x_{-i}) = 0$.

The following condition is needed to guarantee that RUSC is satisfied for all payoffs.

O.7 When consumers are indifferent between the quality/price bundles of two or more firms, any firm whose production of that good results in positive profit must produce as much as desired (up to market demand) before any firm whose production of that good does not result in positive profit.[26] For any $x \in X$ any product t and firm i, denote by J any set of firms such that $f_i^{h^{t,j}}(x) = 0$ for all $j \in J$ and $\varphi_i(x, \Lambda^x) = \varphi_i(x, \Lambda^{x-h^{t,J}}(i))$. Then for all such J, $\theta_j(x, \Lambda^x(j)) = \varphi_j(x, \Lambda^x(j))$ for all $j \in J$, or $\theta_{j'}(x, \Lambda^x(j')) > \varphi_{j'}(x, \Lambda^{x-h^{t,J \smallsetminus j'}}(j'))$ for all $j' \in J$.

O.7 requires that if firm i is indifferent between dominating and being dominated by firm j over product t in the eyes of the consumers, then firm j's profit must be at least what it gets from dominating firm i over product t. It is worth noting that this assumption is trivially satisfied if the cost of production is strictly convex. In fact, this assumption is only relevant if the marginal cost of production is constant for very particular levels of production and

[26] In other words, if a firm's price equals marginal cost (which is locally constant), that firm has lowest priority for consumers.

the price is set to exactly that cost of production. Therefore, while this is a very technical assumption with real restriction, in application it can be irrelevant.

Consider the class of payoff functions $\mathcal{U}$ to be all sharing rules defined by assumptions O.1-O.7.

Proposition 7 *Suppose that O.1-O.7 are satisfied. Then the Multiproduct Oligopoly has an invariant mixed strategy equilibrium, $\widetilde{IE}(\mathcal{U}) = \widetilde{EQ}(\mathcal{U}) \neq \emptyset$.*

Proof. To begin, note that O.7 is sufficient to guarantee $\mathcal{U}$ only includes sharing rules that are RUSC. Take any x^* and $(u^*, x^*) \in \text{cl}G$, if there is a discontinuity at x^*, then O.7 guarantees that $u(x^*) \not\leq u^*$. Thus, $\mathcal{U} = \mathcal{U}_\Psi$. O.6 makes it innocuous to restrict each firm to picking prices in a compact and convex subset of $P \subset \mathbb{R}^T_+$, where for all firms i, $X_i = P \times [0,1]^T$.

Next we show that O.1-O.6 imply A.1-A.3, A.5 and A.6 and thus the proof follows directly from Proposition 2. Assumptions O.1-O.3 directly imply A1-A.3. O.5 gives the specific order partial order for Assumption A.5. Finally, O.6 implies the boundary condition A.6. ■

7 Appendix

The following example formally demonstrates that $\overline{\Pi}(\mu)$ can be different than $\int \overline{\pi}(x) d\mu$.

Example 6 *Consider $N = \{1,2\}$, $X_1 = X_2 = [0,1]$, and $\mathcal{U} = \{u\}$, where u is defined by*

$$\begin{aligned} u_1(x) &= \begin{cases} 1 & \text{if } |x_1 - x_2| > 1/2 \\ 0 & o.w. \end{cases}, \\ u_2(x) &= 1. \end{aligned}$$

Then note that

$$\overline{\pi}_1(x) = \begin{cases} 1 & \text{if } |x_1 - x_2| \geq 1/2 \\ 0 & o.w. \end{cases}.$$

However, consider the probability measures μ defined by

$$\begin{aligned} \mu_1(\{1/2\}) &= 1 \\ \mu_2(\{0\}) &= \mu_2(\{1\}) = 1/2. \end{aligned}$$

Let $\mu^k \to \mu$ *and note that,*

$$\begin{aligned} \lim_k u\left(\mu^k\right) &= \lim_k \Pr\left(|x_1 - x_2| > 1/2\right) \\ &\leq \lim_k \mu_1^k\left([0,1/2]\right) \lim_k \mu_2^k\left([1/2,1]\right) \\ &\quad + \lim_k \mu_1^k\left([1/2,1]\right) \lim_k \mu_2^k\left([0,1/2]\right), \end{aligned}$$

where these limits can be made to exist by choosing a suitable subsequence. Since the for any closed set E, $\lim_k \mu_i^k(E) \leq \mu_i(E)$, *then we have*

$$\begin{aligned} \lim_k u\left(\mu^k\right) &\leq \mu_1\left([0,1/2]\right)\mu_2\left([1/2,1]\right) \\ &\quad + \mu_1\left([1/2,1]\right)\mu_2\left([0,1/2]\right) \\ &= 1/2. \end{aligned}$$

It follows that $\overline{\pi}_1(\mu) \leq 1/2$. *Finally, note that*

$$\begin{aligned} \int \overline{\pi}_1(x) d\mu &= \left(\overline{\pi}_1(1/2,0) + \overline{\pi}_1(1/2,1)\right)/2 \\ &= 1. \end{aligned}$$

Therefore, $\overline{\pi}_1(\mu) < \int \overline{\pi}_1(x) d\mu$.

The following example of a contest with spillovers is a numerical example of a three player auction in which each player's bid impacts the others' payoffs when they win. The model is adapted from the two player model studied by Baye et. al. (2012). In this example, SPM is satisfied, while USPM is violated.

Example 7 *Consider a three player auction in which the winner must pay the sum of all bids while the losers only pay their own bids. Let* $N = \{1,2,3\}$ *and* $X_1 = X_2 = X_3 = \mathbb{R}$. *Define the payoff function* u *as follows.*

$$u_i(x) = P_i(x)(1 - x_1 - x_2 - x_3) - (1 - P_i(x))x_i,$$

where $P_i(x)$ *is the probability that player* i *wins, defined by*

$$P_i(x) = \begin{cases} 1 & \text{if } x_i > \max_{j \neq i} x_j \\ \alpha_i(x) & \text{if } x_i = \max_{j \neq i} x_j \\ 0 & \text{if } x_i < \max_{j \neq i} x_j \end{cases},$$

where $\alpha_i : X \to [0,1]$ *is measurable,* $\alpha_1(x) + \alpha_2(x) + \alpha_3(x) = 1$, *and* $\alpha_i(x) > 0$ *only if* $x_i = \max_{j \neq i} x_j$. *This defines a class of payoff functions* $\mathcal{U}$ *where each* $v \in \mathcal{U}$ *is differentiated by the specification of the tie breaking rule* α.

It can be shown through considerable effort that this game satisfies SPM, while it is easy to see that the game does not satisfy USPM. The reason is that the direction of the optimal deviation when tied with another player depends on the bid of the third player. As such, a single directional deviation is not optimal against every strategy profile of the other players. This highlights the greater generality of SPM, which is still satisfied because it conditions on the strategy profiles that are being chosen.

While the verification of SPM requires extensive computations, the intuition is straightforward. If the expected payoff of winning conditional on a tie is greater than the expected payoff of losing (because the third player's bids are low), then a deviation to a slightly higher bid is used to match $\overline{\pi}$*, as this would result in a win instead of a tie. Otherwise, a deviation to a slightly lower bid is used to match* $\overline{\pi}$*, as this results a loss instead of a tie. To formally verify SPM, these deviations need to be accommodated at all points in the support of a player's strategy.*

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