

# Simple Sufficient Conditions for Weak Reciprocal Upper Semi-Continuity in Extended Games

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**Abstract.** We provide sufficient conditions for a game with discontinuous payoffs to be weakly reciprocally upper semi-continuous in mixed strategies. These conditions are imposed on the individual payoffs and not on their sum, and they can be readily verified in a large class of games even when the sum of payoffs in such games is not upper semi-continuous. We apply our result to establish the existence of mixed strategy equilibria in probabilistic voting competitions where candidates have very general utility functions as well as heterogenous beliefs about the distribution of the voters.

**Key words.** Better reply security, extended games, Weak reciprocal upper semi-continuity, Nash equilibria, Probabilistic voting models.

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## 1. Introduction

In a seminal paper, Reny [11] identified two conditions, payoff security and reciprocal upper semi-continuity, which together play an important role in establishing the existence of Nash equilibria in games with discontinuous payoffs. Unfortunately, these conditions can be difficult to verify in mixed strategies. Recent advances (see [9] and [2]) have allowed for straightforward verification of payoff security in mixed strategies in certain types of games. Furthermore, the condition of reciprocal upper semi-continuity was relaxed in [3] where the notion of weak reciprocal upper semi-continuity was introduced. However, the verification of weak reciprocal upper semi-continuity - and hence the verification of reciprocal upper semi-continuity- in mixed strategies remains a challenge, particularly when the sum of the payoffs is not upper semi-continuous. This paper addresses this problem, providing simple -in a sense we will make precise shortly- sufficient conditions for weak reciprocal upper semi-continuity in mixed strategies. These conditions can be combined with the results in [9], [2], and [3] to establish the existence of mixed strategy equilibrium in a large class of games with discontinuous payoffs.

Specifically, we consider a game  $G$  with a finite number of players with possibly discontinuous payoffs defined on a compact strategy set  $X$ . Let  $\tilde{G}$  denote the extension of  $G$  to mixed strategies, and let  $\Gamma$  and  $\tilde{\Gamma}$  respectively denote the graphs of  $G$  and  $\tilde{G}$ . The notion of better reply security of  $\tilde{G}$  in mixed strategies (better reply security in pure strategies + quasi-concavity of  $G$ ) introduced in [11] ensures that the set of Nash equilibria of  $\tilde{G}$  ( the set of pure strategy equilibria of  $G$ ) is non-empty and closed in  $\mathcal{M}(X)$  (in  $X$ ) and that limits of  $\varepsilon$ -equilibria - as  $\varepsilon$  goes to zero- are also equilibria. In order to establish that  $G$  and  $\tilde{G}$  are better reply secure, one often needs to establish that these games are weak reciprocal upper semi-continuous (henceforth WRUSC).<sup>1</sup> This is a condition that is imposed on the points that are in the topological frontier of the game given by  $Fr(\tilde{\Gamma}) = cl\tilde{\Gamma} \setminus \tilde{\Gamma}$  (resp.  $Fr(\Gamma) = cl\Gamma \setminus \Gamma$ ) [3]. On one hand, this condition is convenient because it is essentially ordinal - in the sense that it is preserved under strictly increasing and continuous transformations-, and it does not impose any requirement on the sum of the payoff functions as in [6] or on the aggregator function of the game as in [10].<sup>2</sup> On the other hand, verifying this condition directly -particularly when the

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<sup>1</sup>The notion of WRUSC can be combined with the notion of payoff security to establish better reply security [3].

<sup>2</sup>See the discussion on page 1030 in [11] on the significance of avoiding imposing condition on the sum of the payoffs in the game. For the proof that the property  $G$  is WRUSC is essentially ordinal, see A in the Appendix.

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sum of the payoffs is not upper semi-continuous (henceforth usc)- can be rather complicated since it requires knowing the boundary set of  $\tilde{\Gamma}$  in  $\mathcal{M}(X) \times \mathbb{R}^I$ .<sup>3</sup>

In this paper, we provide simple conditions that imply that  $\tilde{G}$  is WRUSC. These conditions do not require computing  $Fr(\tilde{\Gamma})$ , and they can be verified even when the sum of the payoffs fails to be usc. These conditions are simple in the following sense: i) they are imposed directly on the individual payoff functions of the game  $G$ , ii) they are preserved under strictly increasing transformations that are continuous, iii) they only require inspecting the behavior of the payoff functions of  $G$  over points of upper discontinuity, which are - in many applications- very small sets with a very simple structure. Our result is particularly easy to apply when all the payoff functions in  $G$  have the same set of upper discontinuities. This covers a large class of games where the upper discontinuity in the payoff arises from a symmetric tie-breaking rule. Furthermore, our result allows us to construct an example where  $\tilde{G}$  is WRUSC but  $G$  is not. This is in contrast to the fact that the stronger notion of reciprocal upper semi-continuity of  $\tilde{G}$  implies the reciprocal upper semi-continuity of  $G$  (see page 1043 in [11]). Finally, we apply our result to establish the better reply security of games of voting games with spatial competition. This, in turn, allows us to establish the existence (and the stability) of equilibria in such games under conditions that are considerably weaker than the current existence results in the literature on voting games.

Our result is based on three observations: First, for every  $(\mu^*, \alpha^*) \in Fr(\tilde{\Gamma})$ , there exists some player  $j$  such that either the expected payoff of this player with respect to  $\mu^*$  is strictly large than  $\alpha_j^*$ , or  $\mu^*$  must assign non-zero probability to the some points of discontinuity of the payoff of this player (Lemma 1 in Section 3). Second, the essential role of various notions of reciprocal upper semi-continuity is to relate the upper discontinuity points of individual payoff functions to the upper discontinuity points of the sum of the payoff functions (Lemma 2 in Section 3). Third, for any player with a given expected payoff, deviations in pure strategies can be expressed as deviations in mixed strategies with the same expected payoff.<sup>4</sup>

## 2. Preliminaries

Consider a game with a finite number of players indexed by the set  $I$ . Each player has a strategy

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<sup>3</sup>If the sum of the payoffs is upper semi-continuous, then both  $G$  and  $\tilde{G}$  are reciprocally upper semi-continuous, and therefore they are also WRUSC.

<sup>4</sup>Mathematically speaking, this a consequence of Theorem 39C in [7].

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set  $X_i$  that is a compact subset of a metric space. Let  $X = \prod_{i \in I} X_i$ . Let  $\mathcal{M}(X_i)$  be the space of Borel probability measures on  $X_i$  equipped with the standard weak topology, and let  $\mathcal{M}(X) = \prod_{i \in I} \mathcal{M}(X_i)$ . For every player  $i$ , the payoff is a Borel measurable bounded function  $U_i : X \rightarrow \mathbb{R}$ . The upper closure of  $U_i$ , denoted by  $\overline{U}_i$ , is defined as

$$\overline{U}_i(x) = \inf_{W \in \mathcal{W}(x)} \sup_{x' \in W} U_i(x'),$$

where  $\mathcal{W}(x)$  is the collection of open sets in  $X$  containing  $x$ .

Recall the following properties of  $\overline{U}_i$ : The function  $\overline{U}_i$  is usc with  $\overline{U}_i \geq U_i$ . Moreover, for any  $x \in X$ , there exist  $x_n \rightarrow x$  such that  $\lim_n U_i(x_n) = \overline{U}_i(x)$ . If  $U_i(x') \geq \alpha$  for all  $x' \in W$  for some  $W \in \mathcal{W}(x)$ , then  $\overline{U}_i(x) \geq \alpha$ .

Let  $D_i$  be the set of points in  $X$  where  $U_i$  is discontinuous, and let  $D_i^{usc}$  be the set of points in  $X$  where  $\overline{U}_i > U_i$ .<sup>5</sup> Clearly  $D_i^{usc} \subseteq D_i$ , and in many applications  $D_i^{usc}$  is actually much smaller than  $D_i$ . Finally, for every  $x_i \in X_i$ , we define  $D_i(x_i) = \{x_{-i} \in X_{-i} | U_i \text{ is discontinuous}\}$ .

Given a game  $G = (X_i, U_i, I)$ , we denote by  $\tilde{G} = (\mathcal{M}(X_i), EU_i, I)$  an extended game that is played on  $\mathcal{M}(X)$ , and the payoff of player  $i$  in the extended game is

$$EU_i(\mu_i, \mu_{-i}) = \int_{X_i} \int_{X_{-i}} U_i d\mu_i d\mu_{-i}.$$

The extended game  $\tilde{G} = (\mathcal{M}(X_i), EU_i, I)$  has an equilibrium  $\mu^* \in \mathcal{M}(X)$ , if for every  $i$ ,

$$EU_i(\mu_i^*, \mu_{-i}^*) \geq EU_i(\mu_i', \mu_{-i}^*), \quad \forall \mu_i' \in \mathcal{M}(X_i).$$

**Definition 1.**  $\tilde{G}$  is WRUSC, if for any  $(\mu^*, \alpha^*) \in Fr(\tilde{\Gamma})$ , there exists a player  $j$  and  $\mu_j \in \mathcal{M}(X_j)$  such that

$$EU_i(\mu_j, \mu_{-j}^*) > \alpha_j^*.$$

## 2. The main result

We start with a lemma that clarifies the key property of the points in  $Fr(\tilde{\Gamma})$ .

**Lemma 1** If  $(\mu^*, \alpha^*) \in Fr(\tilde{\Gamma})$ , then there exists a player  $j$  such that either

a1)  $\mu^*(D_j^{usc}) \neq 0$

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<sup>5</sup>For every  $i \in I$ , the set  $D_i^{usc}$  is Borel measurable. See B. in Appendix for the proof.

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or

b1)  $\alpha_j^* < EU_j(\mu_j^*, \mu_{-j}^*)$ .

**Proof.** Let  $(\mu^*, \alpha^*) \in Fr(\tilde{\Gamma})$ . Let  $(\mu_1^n, \dots, \mu_I^n) \rightarrow (\mu_1^*, \dots, \mu_I^*)$  such that  $\lim_n \int_X U_i(x) d\mu^n = \alpha_i^*$ . First, note that for every  $i$ , the set  $D_i^{usc}$  is Borel measurable (See Appendix B). Suppose  $\mu^*(D_i^{usc}) = 0$  for every  $i$ . Then, for each  $i$ ,  $U_i$  is usc  $\mu^*$ -almost-everywhere on  $X$ . Hence,

$$\alpha_i^* \leq \limsup_n \int_X U_i d\mu^n \leq \int_X U_i d\mu^*,$$

and since  $(\mu^*, \alpha^*) \notin Fr(\tilde{\Gamma})$ , this implies the existence a player  $j$  such that

$$\alpha_j^* < \int U_j d\mu^*.$$

□

We now introduce a notion of upper discontinuity of a game  $G$  at a point in  $X$ .

**Definition 2.** For every  $x \in X$ , let  $\bar{U}(x)$  denote the vector  $(\bar{U}_1(x), \dots, \bar{U}_I(x))$  in  $\mathbb{R}^I$ . The point  $x \in X$  is called a point of reciprocal upper discontinuity of  $G$ , if  $(x, \bar{U}) \notin cl\Gamma$ . We denote that collection of all such points by  $\Psi$ .

It should be clear that if  $U_i$  is continuous for every  $i$ , then the set  $\Psi$  is empty. However, it is possible that  $D_i^{usc}$  is empty for every  $i$  (i.e. all the payoffs are usc), and yet  $\Psi$  is not empty. In most of the applications that we have in mind, the set  $\Psi$  can be obtained by inspection using the following fact:  $x \in \Psi$ , if and only if for any  $x_n \rightarrow x$ , there is some player  $j$  such that either  $\lim_n U_j(x_n)$  does not exist or  $\lim_n U_j(x_n) < \bar{U}_j(x)$ .

The following theorem is our main result:

**Theorem 1** Assume  $G$  satisfies the following conditions for every  $i$ :

a) there exists a sequence of Borel measurable functions  $T_i^m : X_i \rightarrow X_i$  such that for every  $(x_i, x_{-i}) \in X$ , we have

$$\liminf_m U_i(T_i^m(x_i), x_{-i}) \geq \bar{U}_i(x_i, x_{-i}).$$

b)  $D_i^{usc} \subseteq \Psi$

Then, the extended game  $\tilde{G}$  is WRUSC.

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Before we prove Theorem 1, we make the following remarks:

Remark 1: In Theorem 1, the same sequence  $T_i^m$  must work for any  $x_{-i}$ . Furthermore, it is crucial that  $T_i^m$  are Borel measurable, but there are no continuity or convergence assumptions on these functions.

Remark 2: In some games, the functions  $T_i^m$  can be very simple. Consider a player  $i$  and suppose for every  $\varepsilon > 0$ , there exists  $x^\varepsilon$  such that for every  $x \in X$  there exists  $W \in \mathcal{W}(x)$  -a neighborhood of  $x$ - such that

$$U_i(x_i^\varepsilon, x_{-i}) \geq U_i(x') - \varepsilon,$$

for every  $x' \in \mathcal{W}(x)$ . Then,

$$U_i(x_i^\varepsilon, x_{-i}) \geq \overline{U}_i(x) - \varepsilon,$$

and this player satisfies condition (a) of Theorem 1. More specifically, for any  $\varepsilon^m \searrow 0$ , we can take  $T_i^m(x_i) \equiv x^{\varepsilon^m}$ .

To illustrate Theorem 1, we consider the following simple example:

**Example 1** For  $i \in \{1, 2\}$  and  $X_i = [0, 1]$  consider a game with the following payoffs.

$$U_1(x_1, x_2) = \begin{cases} x_1 & \text{if } x_1 < 1/2 \text{ and } x_2 = 1/2 \\ 0 & \text{otherwise} \end{cases} \quad (5)$$

and

$$U_2(x_2, x_1) = \begin{cases} x_2 & \text{if } x_1 = 1/2 \text{ and } x_2 < 1/2 \\ 0 & \text{otherwise} \end{cases} \quad (6)$$

Clearly,  $D_1^{usc} = D_2^{usc} = \{(1/2, 1/2)\}$ . Note that, for every player, the set  $D_i$  is much larger than the set  $D_i^{usc}$ .<sup>6</sup> Moreover, the sum of the payoffs is not usc. In fact,

$$\overline{U}_1(1/2, 1/2) + \overline{U}_2(1/2, 1/2) = 1$$

whereas

$$\overline{U}_1 + \overline{U}_2(1/2, 1/2) = 1/2,$$

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<sup>6</sup>In this example,  $D_1 = \{(x_1, x_2) | x_1 \in (0, 1/2], x_2 = 1/2\}$  and  $D_2 = \{(x_1, x_2) | x_1 = 1/2, x_2 \in (0, 1/2]\}$ .

and

$$U_1(1/2, 1/2) + U_2(1/2, 1/2) = 0.$$

More importantly, and since  $(1/2, 1/2, 1/2, 1/2) \notin \text{cl } \Gamma$ , it is clear that  $(1/2, 1/2)$  is in  $\Psi$ , and therefore condition (b) of Theorem 1 holds. Furthermore, let  $\varepsilon^m \searrow 0$ . Define

$$T_1^m(x_1) = \begin{cases} x_1 & x_1 \neq 1/2 \\ 1/2 - \varepsilon^m & \text{otherwise} \end{cases}$$

$$T_2^m(x_2) = \begin{cases} x_2 & x_2 \neq 1/2 \\ 1/2 - \varepsilon^m & \text{otherwise} \end{cases}$$

Then,  $\lim_m U_1(T_1^m(x_1), x_2) = U_1(x_1, x_2) = \bar{U}_1(x_1, x_2)$  when  $(x_1, x_2) \neq (1/2, 1/2)$ , and  $\lim_m U_1(T_1^m(1/2), 1/2) = \bar{U}_1(1/2, 1/2) = 1/2$ .

Similarly,  $\lim_m U_2(T_2^m(x_2), x_1) = 1/2 \geq \bar{U}_2(x_2, x_1)$  for all  $(x_1, x_2) \in [0, 1] \times [0, 1]$ .

Therefore, condition (a) of Theorem 1 also holds, and this game is WRUSC in mixed strategies.

Remark 3: The game of Example 1 is not WRUSC in pure strategies. In fact, we have the point  $(1/2, 1/2, 1/2, 1/2)$  in  $Fr(G)$  but, for any  $i \in \{1, 2\}$ , it is not possible to find a points  $x_i$  such that  $U_i(x_i, 1/2) > 1/2$ . Therefore, this example leads to the following observation: As noted in [11], if  $\tilde{G}$  is RUSC (in mixed strategies), then so is  $G$  (in pure strategies). For WRUSC, this is no longer true as Example 1 demonstrates.

Remark 4: It might be worth noting that each player in the game of example 1 has an  $\varepsilon$ -weakly dominant strategy for every  $\varepsilon > 0$ . In other words, for every  $i$  and every  $\varepsilon > 0$ , there exists  $x_i^\varepsilon$  such that

$$U_i(x_i^\varepsilon, x_{-i}) \geq U_i(x_i, x_{-i}) - \varepsilon,$$

for every  $x_i \in X_i$  and every  $x_{-i} \in X_{-i}$ . However, no player in this game has a weakly dominant strategy. More generally, we can show that if player in a game has an  $\varepsilon$ -weakly dominant strategy for every  $\varepsilon > 0$ , then this player must satisfy condition (a) of Theorem 1.<sup>7</sup>

The proof of Theorem 1 requires two more lemmas. Our first lemma relates the upper discontinuity points of each  $U_i$  to the upper discontinuity points of their sum.

<sup>7</sup>The example in section 3 shows that there are many games where the players satisfy (a) of Theorem 1 but they do not have any  $\varepsilon$  dominated strategies.

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**Lemma 2.** Let  $D_{\Sigma}^{usc} = \{x \in X \mid \sum_{i \in I} \overline{U}_i > \overline{\sum_{i \in I} U_i}\}$ . For every  $i$ ,  $D_i^{usc} \subseteq \Psi$  implies that  $D_i^{usc} \subseteq D_{\Sigma}^{usc}$ .

**Proof.** WLOG, let  $x \in D_1^{usc} \subseteq \Psi$ . Suppose there exists  $x_n \rightarrow x$  such that

$$\lim_n \sum U_i(x_n) = \overline{\sum U_i(x)}.$$

Note it is always possible to find such a sequence due to the definition of the upper closer of a function. Due to the compactness of  $X$  and the fact the payoff functions are bounded, we can assume -without loss of generality and through the extraction of subsequences if necessary - that

$$\lim_n U_i(x_n) = A_i,$$

where  $A_i \leq \overline{U}_i(x)$ . The fact that  $D_1^{usc} \subseteq \Psi$  implies that  $x \in \Psi$ , which implies that it is not possible to have  $A_i = \overline{U}_i(x)$  for all  $i \in I$ . Therefore, we must have

$$\sum \overline{U}_i(x) > \sum A_i = \overline{\sum U_i(x)},$$

and  $x \in D_{\Sigma}^{usc}$ . □

**Lemma 3** Assume that for every  $i$ ,  $D_i^{usc} \subseteq \Psi$ . Then, for every  $(\mu^*, \alpha^*) \in Fr(\tilde{G})$ , there exists  $j \in I$  such that

$$\alpha_j^* < \int_X \overline{U}_j d\mu^*.$$

**Proof.** Let  $(\mu^*, \alpha^*) \in Fr(\tilde{\Gamma})$ . Let  $(\mu_1^n, \dots, \mu_I^n) \rightarrow (\mu_1^*, \dots, \mu_I^*)$  such that  $\lim_n \int U_i(x) d\mu^n = \alpha_i$ . If (b1) of Lemma 1 holds, then there is nothing to prove. Therefore, we can assume that (a1) of Lemma 1 holds, and there exists a player  $j$  such that

$$\mu^*(D_j^{usc}) \neq 0. \tag{1}$$

Note that for any  $i \in I$ , we always have

$$\sum_i U_i \leq \overline{\sum_i U_i} \leq \sum_i \overline{U}_i,$$

and therefore for every  $n$ , we have

$$\int_X \sum_i U_i d\mu^n \leq \int_X \overline{\sum_i U_i} d\mu^n$$

and

$$\sum_i \alpha_i^* = \lim_n \int_X \sum_i U_i d\mu^n \leq \limsup_n \int_X \overline{\sum_i U_i} d\mu^n \leq \int \overline{\sum_i U_i} d\mu^*$$

where the last inequality holds because  $\overline{\sum_i U_i}$  is usc on  $X$ . Hence, we have

$$\sum \alpha_i^* \leq \int_X \overline{\sum_i U_i} d\mu^*. \quad (2)$$

Lemma 2 and (1) now imply that

$$\int_{D_j^{usc}} \overline{\sum_i U_i} d\mu^* < \int_{D_j^{usc}} \sum_i \bar{U}_i d\mu^*.$$

and therefore

$$\int_X \overline{\sum_i U_i} d\mu^* < \int_X \sum_i \bar{U}_i d\mu^*. \quad (3)$$

Using (2) and (3), we obtain

$$\sum_{i \in I} \alpha_i^* = \int_X \sum_{i \in I} \alpha_i^* d\mu^* < \sum_{i \in I} \int_X \bar{U}_i d\mu^*,$$

and the conclusion of the lemma follows immediately.  $\square$

We now can prove Theorem 1.

**Proof of Theorem 1.** Let  $(\mu^*, \alpha^*) \in Fr(\tilde{\Gamma})$ . Let  $(\mu_1^n, \dots, \mu_I^n) \rightarrow (\mu_1^*, \dots, \mu_I^*)$  and  $\lim_n \int U_i(x) d\mu^n = \alpha_i^*$ . By assumption (b) and Lemma 1, there exists  $j \in I$  such that

$$\alpha_j^* < \int_X \bar{U}_j d\mu^*. \quad (4)$$

By assumption (a) of our theorem, we have

$$\int_X \liminf_m U_j(T^m(x_j), x_{-j}) d\mu^* \geq \int_X \bar{U}_j(x_j, x_{-j}) d\mu^*$$

Now Fatou's Lemma implies

$$\liminf_m \int_X U_j(T_j^{m_0}(x_j), x_{-j}) d\mu^* \geq \int_X \bar{U}_j d\mu^* \quad (5)$$

Combining (4) and (5), we conclude that there exists  $m_0$  such that

$$\int_X U_j(T_j^{m_0}(x_j), x_{-j}) d\mu^* = \int_{X_{-j}} \int_{X_j} U_j(T_j^{m_0}(x_j), x_{-j}) d\mu_j^* d\mu_{-j}^* > \alpha_j^*.$$

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Define the Borel measure  $\mu_j^{m_0} \in \mathcal{M}_j(X_j)$  as follows: for every Borel set  $E \subset X_j$ ,  $\mu_j^{m_0}(E) = \mu^*(E^{m_0})$ , where  $E^{m_0} = \{x'_j \in X_j | T^{m_0}(x'_j) \in E\}$ . Then by Theorem 19.C in [7],

$$\int_{X_{-j}} \int_X U_j(T_j^{m_0}(x_j), x_{-j}) d\mu_j^* d\mu_{-j}^* = \int_{X_{-j}} \int_{X_j} U_j d\mu_j^{m_0} d\mu_{-j}^*,$$

and hence

$$EU_j(\mu_j^{m_0}, \mu_{-j}^*) = \int_{X_{-j}} \int_{X_j} U_j d\mu_j^{m_0} d\mu_{-j}^* > \alpha_j^*,$$

and the game is WRUSC in mixed strategies. □

Our next theorem shows that conditions in Theorem 1 are essentially ordinal; they are preserved under any continuous and strictly increasing transformation. Let  $V_i$  be an interval that contains the range of  $U_i$  for every  $i \in I$  (recall, the payoffs are assumed to be bounded). For every  $i$ , let  $\varphi_i : V_i \rightarrow \mathbb{R}$  be a continuous and strictly increasing function. For every  $i$ , define a new payoffs  $\hat{U}_i(x) = \varphi_i(U_i(x))$ .

**Theorem 2.** If the game  $G = (X_i, U_i, I)$  satisfies conditions (a) and (b) of Theorem 1, then so does the game  $\hat{G} = (X_i, \hat{U}_i, I)$ .

The proof is in the Appendix.

Remark 5: Using Theorem 1 in [3] and Proposition 3.2 in [11], we can establish that  $\tilde{G}$  is better reply secure (and hence has an equilibrium) by combining Theorem 1 with one of the following conditions that establish the payoff security of  $\tilde{G}$  (taken respectively from [11], [9], and [2]) :

- a2) for every  $i$ ,  $U_i$  is lsc in  $x_{-i}$
- b2)  $G$  is uniformly payoff secure
- c2)  $G$  has the disjoint payoff matching property.

A closer look at the proof of Theorem 1 leads to the following corollary.

**Corollary 1.** In any mixed strategy equilibrium of a game satisfying assumption (a) of Theorem 1, every player assigns zero probability to the set of his upper discontinuity points.

Proof. Suppose a game  $G$  satisfies assumptions (a) of Theorem 1, and suppose the corresponding extended game  $\tilde{G}$  has a equilibrium  $\mu^*$ . Then, we must have  $\mu^*(D_i^{usc}) = 0$  for every  $i \in I$ . To see this, assume that  $\mu_j^*(D_j^{usc}) \neq 0$  for some  $j \in I$ . Then, assumption (a) of Theorem 1 (and following

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the same steps in the proof of Theorem 1) implies the existence of a sequence  $\mu^m \in \mathcal{M}(X)$  such that

$$\lim_m \int_X U_j d\mu_j^m d\mu_{-j}^* = \int_X \overline{U}_j d\mu_j, \mu_{-j}^*.$$

Furthermore,  $\mu_j^*(D_j^{usc}) \neq 0$  and the fact that  $\overline{U}_j > U_j$  on  $D_j^{usc}$  imply

$$\int_{D_j^{usc}} \overline{U}_j d\mu_j^* d\mu_{-j}^* > \int_{D_j^{usc}} U_j d\mu_j^* d\mu_{-j}^*,$$

and hence

$$\int_X \overline{U}_j d\mu_j^* d\mu_{-j}^* > \int_X U_j d\mu_j^* d\mu_{-j}^*,$$

and therefore there exists  $\mu_j^{m_0}$  such that

$$EU_j(\mu_j^{m_0}, \mu_{-j}^*) > EU_j(\mu_j^*, \mu_{-j}^*)$$

contradicting the assumption that  $\mu^*$  is an equilibrium. □

### 3. Equilibria in probabilistic voting models

Probabilistic voting models are Hotelling-type games of spatial competition. In these games, candidates competing in an election simultaneously announce policies in some undimensional policy space that is typically represented by the interval  $[0,1]$ . The candidates face uncertainty regarding the outcome of the election that stems from their uncertainty about the preferences of the voters. Such uncertainty is often formulated in terms of the uncertainty candidates regarding the preferences of the median voter.<sup>8</sup> More specifically, candidates are assumed to be uncertain about the location of the median voter's ideal policy in the interval  $[0,1]$ . This uncertainty can be expressed in terms of functions that assign to each candidate his subjective probability of winning the election conditional on the announced policies by all candidates (see [12], [4], [1]). More specifically, we consider an electoral competition between two candidates index by  $i \in \{1, 2\}$ . We assume that the policy space is  $X_i = [0, 1]$ . For  $i \in \{1, 2\}$ , candidate  $i$  believes the location of the ideal policy of median voter is a random variable with values in  $[0, 1]$  and a continuous cdf  $F_i$ .

Given a pair of announced policies, player  $i$  perceives his winning probability to be given by a function  $\pi_i(x_i, x_{-i})$  where

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<sup>8</sup>The candidate who captures the vote of the median voter wins the election.

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$$\pi_i(x_i, x_{-i}) = \begin{cases} F_i\left(\frac{x_i+x_{-i}}{2}\right) & \text{if } x_i < x_{-i} \\ \gamma_i(x) & \text{if } x_i = x_{-i} = x \\ 1 - F_i\left(\frac{x_i+x_{-i}}{2}\right) & \text{if } x_{-i} < x_i \end{cases} \quad (6)$$

and  $\gamma_i$  is a continuous function on  $[0, 1]$ .

Intuitively speaking,  $\pi_i(x_i, x_{-i})$  represents that probability that the median voter will vote for candidate  $i$ , given that the candidates have announced platforms  $x_i$  and  $x_{-i}$ . This is the probability that the median voter's ideal position is closer to  $x_i$  than to  $x_{-i}$ . To see how  $\pi_i$  and  $F_i$  can be derived from more "primitive" assumptions on the voters and their preferences, see chapters 2,3 and 4 in [12] and the examples in [4]. The function  $\gamma_i$  represents a tie breaking rule when both candidates announce the same policy. We assume that  $F_1$  and  $F_2$  have the same median located in  $(0, 1)$ , and we denote this common median point by  $x^c$ . We assume that, for every  $i \in \{1, 2\}$ ,  $\gamma_i$  satisfies the following conditions:

a3)  $\gamma_i(x^c) = F_i(x^c) = 1 - F_i(x^c) = 1/2$

b3)  $\gamma_i(z) < \max\{F_i(z), 1 - F_i(z)\}$  for any  $z \neq x^c$ .

These conditions imply that  $\pi_i$  is continuous at any point that does not lie on the diagonal of the square  $[0, 1] \times [0, 1]$  and at the point  $(x^c, x^c)$  on the diagonal. As a special case, we can set  $\gamma_i \equiv 1/2$  on all of  $[0, 1]$  to obtain the most commonly used tie-breaking rule in spatial voting games. Our assumptions of  $\gamma_i$  also allow us to consider a scenario where a candidate believes that voters are biased against him. In other words, a candidate may believe that given identical political platforms other than  $x^c$ , voters will consider non political attributes such as race, religion, gender, physical appearance, and social status, and this in turn will result in increasing the probability of winning for the other candidate.

Furthermore,(6), assumption (a3) and the continuity of  $F_i$  and  $\gamma_i$  imply

$$\bar{\pi}_i(x_i, x_{-i}) = \begin{cases} \pi_i(x_i, x_{-i}) & \text{if } x_i \neq x_{-i} \\ \text{Max}\{F_i(x), 1 - F_i(x)\} & \text{if } x_{-i} = x_i = x. \end{cases} \quad (7)$$

We assume that the payoff of candidate  $i$  is given by

$$U_i(x_i, x_{-i}) = \pi_i(x_i, x_{-i})r_i(x_i, x_{-i}), \quad (8)$$

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where  $r_i$  is continuous on  $[0, 1] \times [0, 1]$ . We assume

a4) for every  $z \in [0, 1]$ ,  $r_i(z, z) \geq 0$ , and  $r_i(z, z) = 0$ , if and only if  $r_{-i}(z, z) = 0$ .

We now have for every  $(x_i, x_{-i}) \in X$ ,

$$\bar{U}_i(x_i, x_{-i}) = \bar{\pi}_i(x_i, x_{-i})r_i(x_i, x_{-i}). \quad (9)$$

Clearly our assumptions on  $r_i$  hold when the objective the candidates is to maximize the probability of winning the elections i.e.  $r_i(x_i, x_{-i}) \equiv 1$  on the unit square. More importantly, our assumptions on  $r_i$  also allow for the candidates to be policy motivated, office motivated, or both. This covers all the examples discussed in [4] and [13] and the voting examples in [10]. For a specific example of a utility function that satisfies all our assumptions, consider a model where candidate  $i$  gets a utility  $v_i(z)$  when policy  $z$  is implemented by the winner of the election regardless who the winner is, and he gets an additional payoff  $K_i > 0$  from actually winning the election. Therefore, the total (expected) payoff candidate  $i$  given a pair  $(x_i, x_{-i})$  of announced platforms is

$$U_i(x_i, x_{-i}) = \pi_i(x_i, x_{-i})[v_i(x_i) + K_i] + (1 - \pi_i(x_i, x_{-i}))v_i(x_{-i}).$$

Now the model can be readily represented as a game where the payoffs of the candidates have the form specified by (8) with

$$r_i(x_i, x_{-i}) = v_i(x_i) - v_i(x_{-i}) + K_i.$$

In this case, our model satisfies the assumptions of Theorem 1. First, we show that assumption (b) is satisfied.

Let  $D^+ = \{z \in [0, 1] | r_1(z, z) > 0\} = \{z \in [0, 1] | r_2(z, z) > 0\}$  (the equality follows from assumption (a4)). Let  $D = \{(z, z) | z \in D^+ \setminus \{x^c\}\}$  ( $D^+ \setminus \{x^c\}$  is  $D^+$  minus the singleton  $x^c$ ). Assumptions (a3), (b3), and (a4) imply that for the payoffs given in (8), we have  $D_1^{usc} = D_2^{usc} = D$ . Consider a point  $(z, z) \in D$ , and let  $x^n$  be a sequence in  $[0, 1] \times [0, 1]$  such that  $x^n \rightarrow (z, z)$  and  $\lim_n U_i(x^n)$  exists for both candidates. In Appendix E, we show that we must have  $\lim_n U_i(x^n) < \bar{U}_i(z, z)$  for some  $i$ . Hence, we have shown that  $(z, z) \in D$  implies  $(z, z) \in \Psi$ , and for all  $i$ , we have  $D_i^{usc} \subseteq \Psi$ .

Second, we show that condition (a) of Theorem 1 holds. For  $i \in \{1, 2\}$ , let  $\varepsilon_i^m$  be sequence of real numbers strictly decreasing to zero such that

$$0 < x^c - \varepsilon_i^m < 1 \text{ and } 0 < x^c + \varepsilon_i^m < 1. \quad (10)$$

---

For any  $z \in [0, 1]$ , define

$$T_i^m(z) = \begin{cases} z + \varepsilon_i^m & \text{if } z \leq x^c \\ z - \varepsilon_i^m & \text{if } z > x^c. \end{cases} \quad (11)$$

Clearly, for any  $z \in [0, 1]$ ,  $T_i^m(z) \rightarrow z$  and

$$\lim_m U_i(T_i^m(x_i), x_{-i}) = \overline{U}_i(x_i, x_{-i}), \text{ for all } x_{-i} \in [0, 1].$$

Moreover, (10) and (11) imply that  $T_i^m : [0, 1] \rightarrow [0, 1]$ , and (11) implies that each  $T_i^m$  is Borel measurable (for each  $m$ ,  $T_i^m$  is upper semi-continuous, and hence it is a Borel measurable functions). Therefore, for  $i \in \{1, 2\}$ ,  $T_i^m$  satisfies the condition (a) of Theorem 1.

We have so far shown that the game satisfies all the conditions of Theorem 1 and therefore it is WRUSC in mixed strategies. Moreover, for any  $(x_1, x_2)$  in the unit square,  $D_1(T_1^m(x_1)) \subseteq T_1^m(x_1)$  and  $D_2(T_2^m(x_2)) \subseteq T_2^m(x_2)$ . Therefore, we have also established the following for every  $(x_i, x_{-i})$ :

- a5) there exists a sequence  $\{T_i^m(x_i)\} \subset X_i$  such that  $\liminf_m U_i(T_i^m(x_i), x_{-i}) \geq U_i(x_i, x_{-i})$ ,
- b5) for all  $i$  and for any  $m, m'$ , we have  $D_i(T_i^m(x_i)) \cap D_i(T_i^{m'}(x_i)) = \emptyset$ .

By Theorem 1 in [2], (a5) and (b5) imply that our game is payoff secure in mixed strategies. This combined with Remark 5, implies that the game is better reply secure in mixed strategies and must have a mixed strategy equilibrium.

Finally, in probabilistic voting game, we often like to know whether or not the platforms of the candidates “converge” in equilibrium (see [1] and [8], [5]) . In other words, we would like to know whether or not there exists a point  $z^* \in [0, 1]$  such that  $(z^*, z^*)$  is played in equilibrium with some strictly positive probability. Our results in this section -when combined with Corollary 1- indicate that such “convergence” is only possible if  $z^* = x^c$  or if  $r_i(z^*, z^*) = 0$ .

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## Appendix

**A.** WRUSC is essentially ordinal, i.e. it is preserved under continuous and strictly increasing transformations.

**Proof.** Consider the game  $G = (X_i, U_i, I)$ , and assume it is WRUSC For every, assume  $\varphi_i : V_i \rightarrow \mathbb{R}$  is strictly increasing continuous function where  $V_i$  is an interval containing the range of  $U_i$  (recall  $U_i$  is bounded). Let  $\psi_i$  be the inverse of  $\varphi_i$  and note that  $\psi_i$  is also continuous. Consider the game  $\hat{G} = (X_i, \hat{U}_i, I)$  where

$$\hat{U}_i(x) = \varphi_i(U_i(x)).$$

Let  $(x, \alpha)$  be a point in  $Fr(\hat{\Gamma})$ . This means there exists  $x_n \rightarrow x$  such that

a6) for all  $i$ ,  $\lim_n \hat{U}_i(x_n) = \alpha_i$ , and

b6) there exists some player  $j$  with  $\alpha_j \neq \hat{U}_j(x)$

Therefore, by applying  $\psi_i$  and  $\psi_j$  to (a6) and (b6) respectively, we have  $\lim_n U_i(x_n) = \psi_i(\alpha_i)$  and  $\psi_j(\alpha_j) \neq U_j(x)$ . Hence,  $(x, \psi(\alpha)) \notin Fr(\Gamma)$  where  $\psi(\alpha) = (\psi_1(\alpha_1), \dots, \psi_I(\alpha_I))$ . Since  $G$  is WRUSC, this implies that there exists a player  $j$  such that  $U_j(x) > \psi_j(\alpha_j)$ , which implies that  $\hat{U}_j(x) > \alpha_j$ , and  $\hat{G}$  is WRUSC. □

**B.** For every  $i$ , the set  $D_i^{usc}$  is Borel measurable.

For every  $r, s \in \mathbb{Q}$ , where  $\mathbb{Q}$  is the set of rational numbers, define

$$A(r) = \{x \in X | \overline{U}_i(x) \geq r\}$$

and

$$B(s) = \{x \in X | U_i(x) \leq s\}.$$

Since  $\overline{U}_i$  is usc,  $A(r)$  is closed, and hence a Borel set. Since  $U_i$  is Borel measurable,  $B(s)$  a Borel set. Therefore,  $C(r, s) = A(r) \cap B(s)$  is a Borel set. Finally,

$$D_i^{usc} = \bigcup_{\{r, s \in \mathbb{Q} | s < r\}} C(r, s),$$

which is a countable union of Borel sets, and hence it is a Borel set. □

**C.** Suppose  $U : X \rightarrow \mathbb{R}$  is bounded and  $\varphi : \mathbb{R} \rightarrow \mathbb{R}$  is an increasing and continuous function. Then,  $\overline{\varphi(U)} = \varphi(\overline{U})$ . □

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**Proof.** Let  $x \in X$  and let  $x_n \rightarrow x$  such that

$$\lim U(x_n) = \overline{U}(x).$$

By the continuity of  $\varphi$ , we have  $\lim \varphi(U(x_n)) = \varphi(\overline{U}(x))$ . By definition,  $\overline{\varphi(\overline{U}(x))} \geq \lim \varphi(U(x_n))$ , and therefore,

$$\overline{\varphi(\overline{U}(x))} \geq \varphi(\overline{U}(x)). \quad (*)$$

Moreover,  $U(x) \leq \overline{U}(x)$ , and since  $\varphi$  is increasing, we have  $\varphi(U(x)) \leq \varphi(\overline{U}(x))$ . By taking the upper closure of the last inequality, we obtain

$$\overline{\varphi(U(x))} \leq \overline{\varphi(\overline{U}(x))} = \varphi(\overline{U}(x)), \quad (**)$$

where the last equality holds because  $\varphi(\overline{U}(x))$  is usc (the composition of a continuous increasing function with an usc function is usc).

Now (\*) and (\*\*) complete the proof of our claim.  $\square$

#### D. Proof of Theorem 2.

The fact that  $\hat{G}$  satisfies (a) of Theorem 1 is immediate from part C in the Appendix. Define  $\hat{\Psi} = \{x \in X | (x, \hat{U}) \notin \text{cl } \hat{\Gamma}\}$  and  $\hat{D}_i^{usc} = \{x \in X | \hat{U}_i(x) < \overline{\hat{U}_i}(x)\}$ .

The strict monotonicity of  $\varphi_i$  and part C in this Appendix imply that  $\hat{D}_i^{usc} = D_i^{usc}$ .

Therefore, in order to show that (b) of Theorem 1 holds for  $\hat{G}$ , it suffices to show that  $\Psi \subseteq \hat{\Psi}$ . Let  $x \in \Psi$ . Then, for any  $x_n \rightarrow x$  there exists a player  $j$  such that

$$\lim U_j(x_n) < \overline{U}_j(x),$$

and

$$\lim \hat{U}_j(x_n) = \lim \varphi_j(U_j(x_n)) < \varphi_j(\overline{U}_j)(x) = \overline{\varphi_j(U_j(x))} = \overline{\hat{U}_j}(x),$$

where the second equality follows from part C in the appendix. Hence,  $x \in \hat{\Psi}$ .  $\square$

**E.** Condition (b) of Theorem 1 holds for the voting game.

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For  $z \in [0, 1]$  and  $(z, z) \in D$ , assumptions (a3), (b3), and (a4) imply  $z \neq x^c$ ,  $r_1(z, z) > 0$ ,  $r_2(z, z) > 0$ ,  $F_1(z) \neq 1 - F_1(z)$ ,  $F_2(z) \neq 1 - F_2(z)$ ,  $\gamma_1(z) < \text{Max}\{F_1(z), 1 - F_1(z)\}$  and  $\gamma_2(z) < \text{Max}\{F_2(z), 1 - F_2(z)\}$ . Fix some  $i \in \{1, 2\}$ , and let  $x^n$  be any sequence in the unit square converging to  $(z, z)$  with  $\lim U_i(x^n) = \bar{U}_i(z, z)$ . Now (7), (8) and (9) imply that either  $\bar{U}_i(z, z) = F_i(z)r_i(z, z)$  or  $\bar{U}_i(z, z) = (1 - F_i(z))r_i(z, z)$ . Suppose  $\bar{U}_i(z, z) = F_i(z)r_i(z, z)$  and  $F_i(z) > 1 - F_i(z)$ . This and assumption (a3) imply that  $F_{-i}(z) > 1 - F_{-i}(z)$ , and therefore  $\bar{U}_{-i}(z, z) = F_{-i}(z)r_{-i}(z, z)$ . Moreover,  $\lim U_i(x^n) = F_i(z)r_i(z, z)$  also implies that  $x_i^n < x_{-i}^n$  for large enough  $n$ . Therefore,  $\lim U_{-i}(x^n) = (1 - F_{-i}(z))r_{-i}(z, z) < \bar{U}_{-i}(z, z)$ . Similarly, if  $\bar{U}_i(z, z) = (1 - F_i(z))r_i(z, z)$  and  $\lim U_i(x^n) = \bar{U}_i(z, z)$ , then  $\lim U_{-i}(x^n) = F_{-i}(z)r_{-i}(z, z) < \bar{U}_{-i}(z, z)$ .

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