

Price Competition with Decreasing Returns-to-Scale: A General Model of Bertrand-Edgeworth Duopoly

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Abstract

We present a novel approach to analyzing models of price competition. By realizing price competition as a class of all-pay contests, we are able to generalize the models in which pricing behavior can be characterized, accommodating convex (possibly asymmetric) cost structures and general demand rationing schemes. Using this approach, we identify necessary and sufficient conditions for a pure strategy equilibrium and use them to demonstrate the fragility of deterministic outcomes in pricing games. Consequently, we characterize bounds on equilibrium pricing and profits of all mixed strategy equilibria and examine the effect of demand and supply shifts on those bounds. Our focus on bounds can be motivated by the potential for multiple non-payoff equivalent equilibria, as we identify two types of equilibrium strategies through a derivation of sufficient conditions for uniqueness of equilibrium.

Key words: Price competition, Contest, Demand rationing, Convex costs, Capacity constraints.

1 Introduction

Since the inception of mathematical economics, the determination of prices in markets with very few sellers has been a central subject of inquiry. Edgeworth (1925) moved the understanding of this subject forward by appreciating the impact of consumer rationing and the

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prominence of price indeterminacy, or pricing cycles, in duopoly with decreasing returns to scale.¹ His predictions are in stark contrast to the deterministic outcomes associated with the game theoretic models employed by much of the modern literature on oligopoly theory. In this paper, we build upon the literature that studies Bertrand-Edgeworth (BE) games and formalize a duopoly model allowing for general (possibly asymmetric) production technologies and demand rationing. Using this model, we provide a complete characterization of the pure strategy equilibria of the BE game, classifying all such equilibria as Bertrand (marginal cost pricing) or Cournot (market clearing pricing). We further derive precise conditions under which equilibrium pricing is deterministic. In particular, our results highlight the fragility of pure strategy equilibria in BE games, as we demonstrate that they require either binding capacity constraints, discontinuities in marginal costs, or symmetric constant marginal costs. Even with these characteristics, the equilibrium will not be deterministic unless the capacity constraints fall within a particular range or the discontinuities occur at very precise levels of production. These results suggest that further investigation of mixed strategy pricing is needed. To that effect, we present a characterization of the mixed strategy equilibria of the BE game. As a novel approach to analyzing games of price competition, we convert the BE game into a new form of an all-pay contest. This allows for greater ease of analysis, which we take advantage of in order to examine the effects of changes to demand and supply (costs or capacities) on equilibrium pricing and profits.

The model we employ generalizes all but a select few models in the literature, allowing for convex costs of production and virtually unrestricted demand.² In order to conduct our analysis, we identify some of the abstract properties of the BE game with those of traditional all-pay contests. In the BE game, firms place bids in the form of a price in an attempt to win the larger share of the demand, which goes to the firm with the highest bid (lowest price). There are two fundamental distinctions between the BE game and the traditional all-pay contest. First, the payoff of the losing player (the firm with the highest price) depends on the price of the winner through the rationing of residual demand, while traditionally the losing player's payoff depends only on her committed bid. Second, the payoff of the losing player is non-monotonic in her bid, as a reduction in price increases the quantity demanded, possibly raising profits, while an increased bid in traditional contests merely commits the loser to a greater loss. To begin to tackle these complications, we do not analyze the game in its natural form, but rather convert the payoffs into their corresponding contest structure. We then build upon the techniques developed by Siegel (2009, 2010) to characterize the equilibria of this new type of contest. Our results may thus be viewed as a contribution both to the literature on oligopoly pricing and to the literature on contests.

Following Siegel (2009), our basic approach is to bound the equilibrium bids (prices) and payoffs (profits) of the players in the contest. Unlike the outcome of Siegel's model, the bounds on each firm's equilibrium price depends not on its own profit function, but on

¹Vives (1986, 1993) both provide excellent context for Edgeworth's contribution to oligopoly.

²We require only that the monopoly profit be strictly increasing up to its unique maximizer.

that of its competitor. Furthermore, the lower and upper bounds on the equilibrium profits do not coincide, as it is in general possible for there to exist multiple non-payoff equivalent equilibria. We are able to derive multiple conditions under which the equilibrium is unique, however, this requires strong restrictions on the structure of the game. Nevertheless, in doing so, we are able to better understand the types equilibrium strategies when there is such multiplicity. Given this potential for multiple types of equilibria, it is not possible to demonstrate local comparative statics, as most results can be violated by “switching” equilibrium types. Instead, we examine the impacts of demand and supply shifts on the bounds of equilibrium prices and profits. These shifts can accommodate changes in preferences, rationing, cost, or capacity. We are able to demonstrate that an increase in either market demand or residual demand (through changes to the rationing rule) will weakly increase the bounds on the lowest equilibrium price along with the bounds on profits, however, through example we show that the upper bound on pricing may be reduced. An increase in a firm’s supply weakly decreases the bounds on the lowest equilibrium price along with the bounds on the other firm’s profits. While a general prediction cannot be made for the bounds on the profit of the firm with the supply increase, we are able to identify two countervailing effects: a direct effect through which lower costs or higher capacities enhances profitability, and a competitive effect through which those changes increase the level of competition, driving down prices and profits. We use multiple examples to show that either of these effects may dominate. In particular, we show that the competitive effect can dominate when only one firm’s costs are reduced, and thus a firm may be hurt by reducing its own costs. This has important implications for innovation, the adoption and sharing of technology, as well as the evaluation of efficiency gains from firm mergers.

While the origins of the BE model can be traced back to Edgeworth, his basic insights were first formalized into a game theoretic model by Shubik (1959).³ Shubik focused on understanding the range of pricing in mixed strategy equilibrium and the character of pure strategy equilibria when they exist. Our paper thus builds directly upon Shubik’s work, providing much more general answers to his seminal questions. Since Shubik’s formalization, an extensive literature studying BE games has been formed. Within the models of this literature, firms possess constant marginal costs and demand is rationed according to either the efficient or proportional rationing rule.^{4,5} This class of Bertrand-Edgeworth games

³Before Shubik (1959), Shapley (1957) published an abstract with a description of results derived from a game theoretic model of pricing. Other early seminal contributions to BE competition were made by Beckman (1965), Shapley and Shubik (1969), and Levitan and Shubik (1972).

⁴The efficient rationing rule specifies that the highest value consumers are served by the low price firm while low value consumers are rationed. The proportional rule specifies that a uniform distribution of the consumers are served by the low price firm, resulting in a larger residual demand than the efficient rule.

⁵Almost all of the BE literature assumes that the firms have symmetric, constant marginal cost up to capacity. Deneckere and Kovenock (1996) and Allen et al. (2000) are the notable exceptions. These papers focus on the interesting case in which firms have constant marginal costs that are asymmetric. Additionally, the bulk of this literature further restricts demand to be such that a firm’s monopoly profit is concave in its price. Our analysis is based on the considerably weaker assumption that a firm’s monopoly profit is single

has been used to understand fundamental issues in price determination, including duopoly pricing and capacity investment [Levitan and Shubik (1972), Kreps and Scheinkman (1983), Osborne and Pitchik (1986), Davidson and Deneckere (1986), Allen and Hellwig (1993), Deneckere and Kovenock (1996), Lepore (2009)], sequential pricing [Deneckere and Kovenock (1992), Allen (1993), Allen et al. (2000)], large markets [Allen and Hellwig (1986), Vives (1986), Dixon (1992)], oligopoly [Hirata (2009), De Francesco and Salvadori (2010)], and uncertainty [Reynolds and Wilson (2000), Lepore (2008, 2012), de Frutos and Fabra (2011)]. A consequence of the assumptions made thus far in the literature is that the models fail to accommodate industries in which sales occur prior to production, as the argument that having an inventory of goods leads to approximately constant marginal cost does not apply.

Despite the pertinence of factors such as general production technologies and demand rationing, only incremental progress has been made including these features in a theoretical model. Dixon (1992) considers a model of BE oligopoly with strictly convex costs, deriving conditions for the existence of pure strategy equilibrium.⁶ The most closely related treatment is Yoshida (2006), which characterizes equilibrium pricing in symmetric duopoly with convex cost and efficient rationing.⁷ Progress with the analysis of general models with convex costs has been hindered by theoretical problems with existence of equilibrium (pure or mixed) in this setting. However, we utilize recent advances in the literature on existence of equilibrium in discontinuous games by Bagh (2010) and Allison and Lepore (2014) that allow for the straightforward verification of existence of equilibrium in vast generalizations of BE oligopoly.

In Section 2 we present the model and introduce key notation. In order to establish the bounds on equilibrium prices and payoffs, we define the following preliminary objects. First define the *critical judo price* as the highest price either firm can set to guarantee that the other firm would rather maximize its residual profit than undercut. This terminology is based on the sequential pricing model of judo economics in Gelman and Salop (1983).⁸ The second important price we define is the *critical safe price*, which is the maximum price that either firm can set such that the other firm would earn its max-min profit if it were to undercut.

These two critical prices also relate to our characterization of pure strategy equilibrium. A pure strategy equilibria exists if and only if the critical judo and critical safe price are equal and this price maximizes residual profit for both firms. The results on pure strategy equilibrium are shown in Section 3.

peaked in its price.

⁶Hoernig (2007) provides a thorough treatment of price competition with general cost structure and sharing rules for the classical Bertrand model with no consumer rationing.

⁷Yoshida (2002) provides a similar treatment to Yoshida (2006) for a model with linear demand and quadratic cost.

⁸Gelman and Salop (1983) show that, in a two period sequential game, a single potential entrant can use capacity restriction and judo pricing to induce an unconstrained monopolist to allow entry. The concept of judo economics has been used as a basis to understand the equilibrium of Bertrand-Edgeworth duopoly by Deneckere and Kovenock (1992) and Lepore (2009).

In Section 4 we present the general results on mixed strategy equilibria. The primary characterization of the mixed strategy equilibria of the BE game is that lowest equilibrium price will fall between the critical safe price and critical judo price, and consequently, the expected profits of each firm in *all equilibria* are bounded between its monopoly profit at the critical safe price and the critical judo price. This payoff characterization does not rely on uniqueness of equilibrium and applies to all equilibria.⁹ In the process of establishing the payoff bounds we provide abstract bounds for the range of pricing in all equilibria in the spirit of Shubik (1959). We also provide a light characterization of equilibrium strategies by demonstrating that they must be atomless up to a particular price in the support.

We explore the impact of shifts in market demand, residual demand rationing, and costs in Section 5.

Section 6 covers two special cases of the model that have a unique equilibrium. In the case of identical firms, under concavity assumptions, we are able to prove that the equilibrium is unique. In analyzing the uniqueness of equilibrium in this setting, we are able to clearly identify the two types of equilibria in the general model: one in which firms mix continuously over the support of the equilibrium and another in which the firms have gaps in the supports of their equilibrium strategies. These two equilibrium types can result in different expected payoffs. This is the prohibitive factor for uniqueness in the general game and limits standard comparative statics directly on expected profits. This lends some intuition as to why our general characterization only provides bounds on the equilibrium pricing and profits. The second special case we consider is one in which each firm's residual profit is independent of the other firm's price, which generalizes the BE game with constant marginal cost and efficient demand rationing. In this case, we provide a closed form solution for the unique equilibrium strategies.

We conclude with a discussion in Section 7. The proofs of existence of equilibrium and some technical lemmas are located in the Appendix.

2 The Model

Consider a homogeneous product industry with two firms $i = 1, 2$. We will use j to refer to the firm other than i . The firms simultaneously and independently announce prices. We denote by p_i the price of firm i and by p the vector of both firms' prices. Since p is the vector of prices (p_1, p_2) , we will use x to unambiguously denote a single price when it is not associated with a particular firm. Each firm i has a continuous, nondecreasing, weakly convex cost of production $c_i : \mathbb{R}_+ \mapsto \mathbb{R}_+$ with $c_i(0) = 0$. The market demand $D : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is continuous and nonincreasing.

⁹Technically, all pure strategy equilibria satisfy this characterization, however, such an approach is not needed, as our direct characterization of such outcomes provides far more information.

Each firm i has a capacity k_i that serves as an upper bound on the quantity that can be produced. Thus, the production problem faced by firm i at a price p_i is

$$\max_{z \in [0, k_i]} \pi_i(p_i, z) = p_i z - c_i(z).$$

Let $s_i(p_i)$ denote the largest quantity that solves this optimization problem.¹⁰ The quantity $s_i(p_i)$ may be referred to as firm i 's supply, the maximum quantity that it is willing to produce at any given price. Inherently, $s_i \leq k_i$, so the supply functions account for the capacity constraints. It will be useful to note that the assumptions on c_i imply that $s_i(p_i)$ is nondecreasing and upper semicontinuous, and that $\pi_i(p_i, z)$ is strictly increasing in z for all $z \in [0, \liminf_{x \rightarrow p_i} s_i(x)]$. This further implies that each s_i is continuous from the right.

If $p_i < p_j$, the demand served by firm i is $Q_i(p_i) = \min \{D(p_i), s_i(p_i)\}$. We make minimal assumptions as to which portion of the demand is served by firm i when $p_i < p_j$, only that there is a continuous function $\lambda_i : \{(p_i, p_j) \in \mathbb{R}_+^2 : p_i \leq p_j\} \rightarrow [0, 1]$ which denotes the share of firm i 's quantity that satiates firm j 's demand.¹¹ Note that the function λ_i may depend on which firm i is the low price firm, allowing for the possibility of asymmetric rationing. The residual demand served by firm j is

$$Q_j^r(p) = \min \{ \max \{ D(p_j) - Q_i(p_i) \lambda_i(p), 0 \}, s_j(p_i) \}.$$

This general framework is consistent with the assumption that consumers prefer lower prices, and so the high price firm may sell a positive quantity only if the low price firm exhausts its supply. To highlight the role that λ_i plays in determining the rationing rule, consider two choices for the functions λ_i given by $\lambda_i^e(p) = 1$ and $\lambda_i^p(p) = D(p_j)/D(p_i)$. The rationing rule under λ_i^e is the well known efficient rule, whereas the rule under λ_i^p is the proportional rationing rule.

We define two different indirect profit functions for a firm based on whether the firm has the lower price or higher price. The *front-side profit* of the firm i with a lower price than firm j is

$$\varphi_i(p_i) = p_i Q_i(p_i) - c_i(Q_i(p_i)).$$

On the other hand, the *residual profit* of the firm i with a higher price than firm j is

$$\psi_i(p) = p_i Q_i^r(p) - c_i(Q_i^r(p)).$$

The residual profit ψ_i is undefined for prices such that $p_i < p_j$ since consumers would not shop at firm j before firm i , and thus firm i could not receive its residual profit.

¹⁰This specification of $s_i(p_i)$ follows from Dixon (1987), Maskin (1986), Bagh (2010) and Allison and Lepore (2014).

¹¹A simple way to understand the purpose of λ_i is to consider the case in which a continuum of consumers have unit demand. In this case, λ_i specifies the fraction of the consumers served by firm i that have willingness to pay of at least p_j .

Based on our specifications above, φ_i and ψ_i are continuous in p_i and lower semicontinuous in p_j . We make the following assumptions about the profit functions φ_i and ψ_i .

Assumption 1 φ_i has a unique maximizer \widehat{p}_i with $\varphi_i(\widehat{p}_i) > 0$.¹² φ_i is strictly increasing at any price $p_i < \widehat{p}_i$ such that $\varphi_i(p_i) > 0$.

This assumption on the front-side profit is weaker than assuming strict quasiconcavity of φ_i as it does not restrict behavior at prices $p_i > \widehat{p}_i$.

Assumption 2 For any $p_i \geq 0$, $\psi_i(p_i, p_j)$ is nonincreasing in p_j . For any $p_j \geq 0$, $\psi_i(\widehat{p}_i, p_j) \geq \psi_i(p'_i, p_j)$ for all $p'_i \geq \widehat{p}_i$.

Based on the construction of $Q_i^r(p)$, $\varphi_i(p_i) \geq \psi_i(p_i, p_j)$ for all $p_i \geq p_j$. The following assumption is important for our characterization.

Assumption 3 There exists a price $\underline{\rho} \geq 0$ such that for each firms i ,

$$\begin{aligned} \varphi_i(x) &> \psi_i(x, x) && \text{for all } x \in (\underline{\rho}, \widehat{p}_i], \\ \varphi_i(x') &= \psi_i(x', x') && \text{for all } x' \leq \underline{\rho}. \end{aligned}$$

Note that given Assumption 3, Assumption 1 implies that φ_i is positive and strictly increasing on $(\underline{\rho}, \widehat{p}_i]$.

Remark 1 If each firm's supply function s_i is continuous, then the existence of such a price $\underline{\rho}$ follows from the structure of the demand rationing assumptions and is defined by

$$\underline{\rho} = \sup\{x : s_1(x) + s_2(x) = D(x)\}.$$

To understand why this is the case, note that if both firms set the same price x , then each $\lambda_i(x, x) = 1$. When $x \leq \underline{\rho}$, it must be that $s_1(x) + s_2(x) \leq D(x)$, so there is sufficient demand to satiate total supply, regardless of which firm receives the residual profit, and so both the residual and front-side profit must be identical for each firm. When $x > \underline{\rho}$, it must be that $s_1(x) + s_2(x) > D(x)$, and so there is insufficient demand to satiate total supply. As such, either firm's residual profit must be strictly lower than their front-side profit since they would be selling less than their optimal quantity. Note that if the profit at this restricted quantity is identical to the profit at the optimal quantity, then marginal cost must be constant, which induces a discontinuity in the supply function, and so cannot fit into this case. By making Assumption 3, we allow for cost functions that are either kinked or flat at some points (marginal cost discontinuous or constant) without inhibiting our ability to characterize the equilibria.

¹²Although we have omitted the capacities from our notation, it should be apparent that \widehat{p}_i can vary based on firm i 's capacity.

Remark 2 *Assumption 3 accommodates the standard case of symmetric constant marginal cost up to capacity with prices restricted to be weakly greater than marginal cost. On the other hand, Assumption 3 does not accommodate the case of asymmetric constant marginal costs. In the case that firms have asymmetric constant marginal costs the conditions we use to show existence are violated. This highlights the remarkable contribution of Deneckere and Kovenock (1996) and the constructive method used to show existence in a pricing game with the efficient rationing and asymmetric constant marginal cost.*

Lastly, we make the following assumption to rule out the possibility of a natural monopoly.

Assumption 4 *For each i and j , $\varphi_i(\widehat{p}_j) > 0$.*

Each firm i 's profit is specified as follows

$$u_i(p) = \begin{cases} \varphi_i(p_i) & p_i < p_j \\ \alpha_i(p)\varphi_i(p_i) + (1 - \alpha_i(p))\psi_i(p) & p_i = p_j \\ \psi_i(p) & p_i > p_j \end{cases},$$

where $\alpha_i(p) \in [0, 1]$ and $\alpha_1(p) + \alpha_2(p) \in (0, 2)$. If we instead assume that $\alpha_1 + \alpha_2 = 1$, then this restricts attention to sharing rules that assign one firm its front-side profit and the other its residual profit, with some randomization over the assignment. By permitting the sum of the shares to be greater than one, we allow for any sharing of demand at ties, which can naturally result in each firm receiving a (non-stochastic) profit strictly between its front-side and residual profits. While this could also be accommodated by putting the sharing rule directly on the demand, placing the sharing rule on the profits does not impact the results and actually provides notational parsimony to the equilibrium characterization that greatly enhances the clarity of the exposition.

Note that for each firm i , any price $p_i > \widehat{p}_i$ is always strictly dominated by $p'_i = \widehat{p}_i$. Given Assumptions 1-3, it follows that a price of $p_i = \underline{\rho}$ strictly dominates all prices $x < \underline{\rho}$. We thus restrict prices to the domain $[\underline{\rho}, \widehat{p}_1] \times [\underline{\rho}, \widehat{p}_2]$. The continuity of ψ_i in p_i and the compactness of its domain imply that the residual profit function has a largest maximizer $\widetilde{p}_i(p_j) \leq \widehat{p}_i$. We denote the set of maximizers of ψ_i at any p_j by $\widetilde{P}_i(p_j)$.

Denote the maximized residual profit by $\widetilde{\psi}_i(p_j)$, that is,

$$\widetilde{\psi}_i(p_j) = \max_{p_i \geq p_j} \psi_i(p_i, p_j).$$

Define \bar{r}_j to be firm j 's *judo price*, the highest price that firm j can set to guarantee that firm i would rather maximize its residual profit than undercut. Formally,

$$\bar{r}_j = \sup\{p_j \in [\underline{\rho}, \widehat{p}_j] \mid \varphi_i(p_j) \leq \widetilde{\psi}_i(p_j)\}.$$

Define \underline{r}_j to be firm j 's *safe price*, the highest price such that the front-side profit of firm i equals the highest profit that firm i can guarantee itself. Formally,

$$\underline{r}_j = \sup\{p_j \in [\underline{\rho}, \widehat{p}_j] \mid \varphi_i(p_j) \leq \underline{u}_i\},$$

where $\underline{u}_i = \sup_{p_i} \inf_{p_j} u_i(p_i, p_j) = \max_{x \in [\underline{\rho}, \widehat{p}_j]} \psi_i(x, x)$.

Define the larger of the two firms' judo prices to be *critical judo price*, denoted by $\bar{r} = \max \bar{r}_i$. Similarly, define the larger of the two firms' safe prices to be the *critical safe price*, denoted by $\underline{r} = \max \underline{r}_i$. Based on their definitions, the judo price is always weakly greater than the safe price, that is, $\bar{r} \geq \underline{r}$.

Define firm j 's *judo profit* to be the front-side profit of firm j at the critical judo price, denoted by $\bar{\varphi}_j \equiv \varphi_j(\bar{r})$. Similarly, define firm j 's *safe profit* to be the front-side profit of firm j at the critical safe price, $\underline{\varphi}_j \equiv \varphi_j(\underline{r})$.

For equilibrium strategies $\mu = (\mu_1, \mu_2)$, we use \underline{x}_i and \bar{x}_i to denote the infimum and supremum of the support of firm i 's strategy, respectively. We will use \underline{x} to denote the minimum of \underline{x}_1 and \underline{x}_2 , and \bar{x} to denote the maximum of \bar{x}_1 and \bar{x}_2 .¹³ Further, we define F_i to be the distribution function (*CDF*) of firm i 's mixed strategies on $[\underline{x}, \bar{x}]$, with $F = (F_1, F_2)$. Lastly, we will use $M_i(x)$ to denote the probability that firm i gets the front-side profit when playing the price x in any fixed equilibrium, so that $M_i(x) = \mu_j([\underline{x}, x]) + \alpha_i(x, x)\mu_j(\{x\})$.¹⁴

3 Pure Strategy Equilibria

Our first objective is to understand when price indeterminacy is resolved by equilibrium play. To that end, we present necessary and sufficient conditions for the existence of a (symmetric) pure strategy equilibrium. Under a mild restriction, our conditions become necessary and sufficient for this to be the unique equilibrium. When our conditions fail to hold, all equilibria of the pricing game must be in mixed strategies. We additionally classify all pure strategy equilibria as either Bertrand (akin to marginal cost pricing) or Cournot (a market clearing price). As it turns out, the pure strategy equilibrium requires either capacity constraints, kinked cost functions, or a kinked demand function, and so a smooth game with strictly convex costs will never have pure strategy pricing.

The key aspect that permits a pure strategy equilibrium is that both firms have the same highest price that makes them indifferent between receiving the front-side and residual profits

¹³Here the bounds \underline{x}_i and \bar{x}_i are inherently dependent on the equilibrium strategies, though we suppress notation indicating this for clarity as there is no ambiguity as to which strategies they correspond to.

¹⁴While α_i need not represent the probability of receiving the front-side profit, it is convenient to use that interpretation here.

and this price is a maximizer of the residual profit for both firms. The following proposition demonstrates that this is both necessary and sufficient for the existence of a pure strategy Nash equilibrium.

Proposition 1 *In any pure strategy equilibrium, both firms must set a price of $\underline{\rho}$. This pricing profile is an equilibrium if and only if $\underline{\rho} \in \tilde{P}_i(\underline{\rho})$ for each firm i .*

Intuitively, it is easy to understand why both firms setting a price of $\underline{\rho}$ is the only possible pure strategy equilibrium. If both firms set a price higher than $\underline{\rho}$, then each firm's front-side profit is higher than their respective residual profit. Since both firms cannot receive their front-side profit with certainty, at least one firm has incentive to undercut. Further, no firm would ever set a lower price, as it is possible to increase its price without adjusting its quantity. Therefore, since the front-side profit at a price of $\underline{\rho}$ is equal to the residual profit, then $\underline{\rho}$ being a residual maximizer for both firms is necessary and sufficient for neither firm to possess a profitable deviation to a higher price.

Before we prove Proposition 1, we first prove the following lemma that is instrumental in many of the proofs of this paper. The lemma establishes that relevant ties ($p_1 = p_2 > \underline{\rho}$) occur with probability zero in all equilibria of the BE game.¹⁵ That is, all equilibria are atomless at pricing ties above the price such that front-side profit equal its residual profit.

Lemma 1 *All equilibria are atomless at any $p_1 = p_2 > \underline{\rho}$. Consequently, the equilibrium strategies and payoffs are unaffected by the choice of sharing rule α .*

Proof of Lemma 1. We first show that all equilibria are atomless at any price profile with $p_1 = p_2 > \underline{\rho}$. Let $x > \underline{\rho}$ and suppose that there is an equilibrium μ such that $\mu(\{(x, x)\}) > 0$. By assumption, there is some firm i such that $\alpha_i(x, x) < 1$. Let $\varepsilon > 0$ and consider a sequence of deviations by firm i to $\tilde{\mu}_i^n$ defined by

$$\tilde{\mu}_i^n(E) = \begin{cases} \mu_i(E \cup \{x\}) & \text{if } x - \delta_n \in E \\ \mu_i(E \setminus \{x\}) & \text{otherwise} \end{cases},$$

where each δ_n is chosen so that $0 < \delta_n < 1/n$ and $\mu_j(\{x - \delta_n\}) = 0$. That is, $\tilde{\mu}_i^n$ is the measure created from μ_i by shifting all mass from the price x to the price $x - \delta_n$. Then note that

$$\int u_i(p) d\tilde{\mu}_i^n \times \mu_j = \int u_i(p) d\mu + \mu_i(\{x\}) \int (u_i(x - \delta_n, p_j) - u_i(x, p_j)) d\mu_j.$$

We will show that $\lim_n \int (u_i(x - \delta_n, p_j) - u_i(x, p_j)) d\mu_j > 0$ for sufficiently large n , which will guarantee a profitable deviation for firm i , violating μ_i as an equilibrium strategy. Note

¹⁵Ties at a price $x \leq \underline{\rho}$ are irrelevant since each firm's front-side profit is identical to its residual profit.

that

$$u_i(x - \delta_n, p_j) - u_i(x, p_j) = \begin{cases} \psi_i(x - \delta_n, p_j) - \psi_j(x, p_j) & \text{if } p_j < x - \delta_n \\ \varphi_i(x - \delta_n) - \psi_j(x, p_j) & \text{if } x - \delta_n \leq p_j < x \\ \varphi_i(x - \delta_n) - \alpha_i(x, x)\varphi_i(x) \\ \quad + (1 - \alpha_i(x, x))\psi_i(x, p_j) & \text{if } p_j = x \\ \varphi_i(x - \delta_n) - \varphi_i(x) & \text{if } p_j > x \end{cases}.$$

It follows that the pointwise limit as $n \rightarrow \infty$ is

$$\lim_n (u_i(x - \delta_n, p_j) - u_i(x, p_j)) = \begin{cases} 0 & \text{if } p_j < x \\ (1 - \alpha_i(x, x))(\varphi_i(x) - \psi_i(x, p_j)) & \text{if } p_j = x \\ 0 & \text{if } p_j > x \end{cases}.$$

Thus, since $|u_i| \leq \varphi_i(\widehat{p}_i)$, then by the Lebesgue dominated convergence theorem,

$$\begin{aligned} \lim_n \int (u_i(x - \delta_n, p_j) - u_i(x, p_j)) d\mu_j &= \int \lim_n (u_i(x - \delta_n, p_j) - u_i(x, p_j)) d\mu_j \\ &= \mu_j(\{x\})(1 - \alpha_i(x, x))(\varphi_i(x) - \psi_i(x, p_j)). \end{aligned}$$

By assumption, $\mu_j(\{x\}) > 0$, and since $x > \underline{\rho}$, then $\varphi_i(x) > \psi_i(x, x)$. Therefore, since $\alpha_i(x, x) < 1$, μ_i^n is a profitable deviation for firm i for sufficiently large n , violating μ as an equilibrium. We conclude that μ does not have mass at x .

Next we show that the equilibrium is invariant to the choice of α . Let μ be an equilibrium given the sharing rule α with expected profits $\pi = (\pi_1, \pi_2)$ and consider another sharing rule α' . Let $u_i(x, \mu_j)$ denote firm i 's expected payoff when choosing a price x given α and $u'_i(x, \mu_j)$ the corresponding payoff given α' . To show that μ is an equilibrium for the game with sharing rule α' , it will suffice to show that for each player i , (i) $u'_i(x, \mu_j) = \pi_i$ μ_i -almost everywhere and (ii) $u'_i(x, \mu_j) \leq \pi_i$ for all prices x .

(i) Note that the sharing rule does not influence the payoffs at any price x such that $\mu_j(\{x\}) = 0$, and so $u_i(x, \mu_j) = u'_i(x, \mu_j)$ at all such prices. Further, at all prices $x \leq \underline{\rho}$, $u_i(x, \mu_j) = \varphi_i(x) = u'_i(x, \mu_j)$. The first part of this lemma demonstrates that $\mu_i(\{x\}) = 0$ for all x such that $\mu_j(\{x\}) > 0$. Since μ_j has at most countably many atoms, then $\mu_i(\{x : \mu_j(\{x\}) > 0\}) = 0$. It follows that $u'_i(x, \mu_j) = \pi_i$ μ_i -almost everywhere.

(ii) As we have shown in part (i), $u_i(x, \mu_j) = u'_i(x, \mu_j)$ except possibly at prices $x > \underline{\rho}$ such that $\mu_j(\{x\}) > 0$. Consider any such price x and let $\{x^k\}$ be a sequence such that $x^k \rightarrow x$, $x^k < x$ for all k , and $\mu_j(\{x^k\}) = 0$ for all k . Then note that the continuity of φ_i and ψ_i in p_i imply that $\lim_k u'_i(x^k, \mu_j) \geq u'_i(x, \mu_j)$. Since $\mu_j(\{x^k\}) = 0$ for all k , then $u_i(x^k, \mu_j) = u'_i(x^k, \mu_j)$ for all k . If $u'_i(x, \mu_j) > \pi_i$, then $u_i(x^k, \mu_j) > \pi_i$ for sufficiently large k , violating μ_i as an equilibrium strategy with the sharing rule α . Therefore, $u'_i(x, \mu_j) \leq \pi_i$ for all x .

We conclude that μ is an equilibrium given the sharing rule α' . ■

We now proceed to the proof of Proposition 1.

Proof of Proposition 1. To begin the proof we argue that any pure strategy equilibrium must be symmetric. Suppose to the contrary that there is an asymmetric equilibrium with $p_i^* < p_j^*$. This means firm i gets $\varphi_i(p_i^*)$ with certainty. There are two cases to consider: (i) $p_i^* < \widehat{p}_i$ and (ii) $p_i^* = \widehat{p}_i$. We may ignore the case in which $p_i^* > \widehat{p}_i$ since firm i would trivially be better off with a price of \widehat{p}_i . In case (i), playing $p_i' \in (p_i^*, p_j^*)$ is strictly better for firm i since it would earn a profit of $\varphi_i(p_i') > \varphi_i(p_i^*)$ with certainty. In case (ii), there must be no residual demand for firm j at p_j^* . If there were any residual demand remaining at \widehat{p}_i , then by the continuity of D , firm i could increase its price slightly, sell the same quantity and make strictly greater profit than at \widehat{p}_i , violating \widehat{p}_i as it's front-side profit maximizer. Since there is no residual demand for firm j , it must be that firm j has zero profit. Thus, since Assumption 4 guarantees that $\varphi_j(\widehat{p}_i) > 0$, firm j would be better off charging some price $p_j' < \widehat{p}_i$ in order to guarantee a positive profit.

Next we show that any symmetric strategy profile $(x^*, x^*) \neq (\underline{\rho}, \underline{\rho})$ cannot be an equilibrium. Let (x^*, x^*) be an equilibrium. It follows immediately from Lemma 1 that $x^* \leq \underline{\rho}$. Suppose that $x^* < \underline{\rho}$, yielding a profit of $\varphi_i(x^*) = \psi_i(x^*, x^*)$ for each firm i . Then note that by playing $p_i = \underline{\rho}$, firm i earns a profit of $\psi_i(p_i, x^*) \geq \psi_i(p_i, p_i) = \varphi_i(p_i) > \varphi_i(x^*)$. We conclude that $x^* = \underline{\rho}$.

It remains to be shown that $(\underline{\rho}, \underline{\rho})$ is an equilibrium if and only if $\underline{\rho} \in \widetilde{P}_i(\underline{\rho})$ for each firm i . Note that if $\underline{\rho} \notin \widetilde{P}_i(\underline{\rho})$, then there exists a $p_i > \underline{\rho}$ such that $\psi_i(p_i, \underline{\rho}) > \psi_i(\underline{\rho}, \underline{\rho}) = u_i(\underline{\rho}, \underline{\rho})$. It follows that if $(\underline{\rho}, \underline{\rho})$ is an equilibrium, then $\underline{\rho} \in \widetilde{P}_i(\underline{\rho})$ for each firm i . Further, if $\underline{\rho} \in \widetilde{P}_i(\underline{\rho})$ for each firm i , then for all prices $p_i > \underline{\rho}$, $\psi_i(p_i, \underline{\rho}) < \psi_i(\underline{\rho}, \underline{\rho})$, and so neither firm can increase its profits by increasing its price. Since φ_i is strictly increasing, neither firm can increase its profits by reducing its price. Therefore, if $\underline{\rho} \in \widetilde{P}_i(\underline{\rho})$ for each firm i , then $(\underline{\rho}, \underline{\rho})$ is an equilibrium. ■

Proposition 1 is important in that it specifies the exact circumstances that Edgeworth's concerns about price indeterminacy can be alleviated. But in this general setting, existence of a pure strategy equilibrium does not guarantee uniqueness of this equilibrium. Since $\underline{\rho}$ is uniquely defined, it is the only pure strategy equilibrium candidate, however, it may be that another mixed strategy equilibrium concurrently exists. The following proposition demonstrates that no other equilibrium in pure or mixed strategies may exist as long as $\underline{\rho}$ is the only residual maximizer for each firm when the other sets a price of $\underline{\rho}$.

Proposition 2 *Suppose that a pure strategy equilibrium exists. If $\underline{\rho}$ is the unique maximizer of $\psi_i(p_i, \underline{\rho})$ for each firm i , then both firms pricing at $\underline{\rho}$ is the unique equilibrium of the BE game.*

Proof of Proposition 2. Assume that a pure strategy equilibrium exists. As argued in the previous proof, any price $x < \underline{\rho}$ is strictly dominated, so we need only consider equilibria

in which prices $x' > \underline{\rho}$ are chosen with positive probability. Suppose that there exists an equilibrium in which the support of some firm i 's strategy is such that $\bar{x}_i > \underline{\rho}$. Using Lemma 1, we may without loss of generality assume that $\bar{x}_i \geq \bar{x}_j$ and that firm j 's strategy does not have an atom at \bar{x}_i . Then note that when choosing a price at or near \bar{x}_i , firm i earns a profit of approximately $\int \psi_i(\bar{x}_i, p_j) d\mu_j$. By assumption,

$$\int \psi_i(\bar{x}_i, p_j) d\mu_j \leq \psi_i(\bar{x}_i, \underline{\rho}).$$

If $\underline{\rho} = \max \tilde{P}_i(\underline{\rho})$ for each firm i , then $\psi_i(\bar{x}_i, \underline{\rho}) < \psi_i(\underline{\rho}, \underline{\rho}) = \varphi_i(\underline{\rho})$. This contradicts prices at or near \bar{x}_i as equilibrium strategies. ■

Remark 3 *Based on Proposition 2, it is evident that in a model such that both firms have unique residual profit maximizers, non-degenerate mixed strategy and pure strategy equilibrium cannot coexist for the same parameters. Thus, in this environment, the necessary and sufficient condition for existence of pure strategy equilibrium also guarantees its uniqueness. In getting back to Edgeworth's theme of price indeterminacy, Proposition 2 provides precise conditions for determinant prices in this class of BE game.*

Propositions 1 and 2 provide the basic character of all pure strategy equilibria of the BE game. We turn now to strengthening this characterization by investigating the nature of $\underline{\rho}$ and classifying all pure strategy equilibria of this game as one of two distinct types. The first type of pricing requires a price equals marginal cost condition *a la* Bertrand pricing. The second type of equilibrium requires supply to equal demand with prices above marginal cost *a la* Cournot pricing. These types are defined formally as follows.

$$\text{Type B: } D(\underline{\rho}) \leq \min s_i(\underline{\rho}),$$

$$\text{Type C: } D(\underline{\rho}) = s_1(\underline{\rho}) + s_2(\underline{\rho}).$$

In the following proposition we show that all pure strategy equilibria must be of Type B or C.

Proposition 3 *All pure strategy equilibria are such that either $D(\underline{\rho}) \leq \min s_i(\underline{\rho})$ or $D(\underline{\rho}) = s_1(\underline{\rho}) + s_2(\underline{\rho})$.*

Proof of Proposition 3. Assume that there is a pure strategy equilibrium. Suppose to the contrary that either (i) $D(\underline{\rho}) > s_1(\underline{\rho}) + s_2(\underline{\rho})$, or (ii) $D(\underline{\rho}) < s_1(\underline{\rho}) + s_2(\underline{\rho})$ and $D(\underline{\rho}) > s_i(\underline{\rho})$.

(i) Suppose first that $D(\underline{\rho}) > s_1(\underline{\rho}) + s_2(\underline{\rho})$. By continuity of D and right-continuity of each s_i , it follows that $D(x) > s_1(x) + s_2(x)$ for some $x > \underline{\rho}$. Thus, it must be that $Q_1^r(x, x) = s_1(x)$, and so $\psi_i(x, x) = \varphi_i(x)$, contradicting Assumption 3 that $\psi_i(x, x) < \varphi_i(x)$ for all $x > \underline{\rho}$.

(ii) Suppose next that $D(\underline{\rho}) < s_1(\underline{\rho}) + s_2(\underline{\rho})$ and $D(\underline{\rho}) > s_i(\underline{\rho})$. In this case, $Q_i(\underline{\rho}) = s_i(\underline{\rho}) > D(\underline{\rho}) - s_j(\underline{\rho}) \geq Q_i^r(\underline{\rho}, \underline{\rho})$. Thus, $\varphi_i(\underline{\rho}) > \psi_i(\underline{\rho}, \underline{\rho})$, contradicting Assumption 3. ■

As mentioned previously, Type B pricing has the flavor of Bertrand pricing and Type C has the flavor of Cournot. Now we address two special cases where this pricing is exactly that of Bertrand or Cournot.

Consider the special case in which each firm has a strictly convex cost function c_i . The following proposition demonstrates that in this case, there is no possibility of Bertrand pricing in equilibrium.

Proposition 4 *If each firm has strictly convex cost c_i , then the only possible pure strategy pricing equilibria are Type C.*

Proof of Proposition 4. Let each c_i be strictly convex and suppose to the contrary that there is a pure strategy equilibrium with $D(\underline{\rho}) \leq \min s_i(\underline{\rho})$. It follows that there is no residual demand if one firm sets a price of $\underline{\rho}$ and the other sets a price $x > \underline{\rho}$. Thus, $\psi_i(\underline{\rho}, \underline{\rho}) = 0$ for both firms i . By definition of $\underline{\rho}$, $\varphi_i(\underline{\rho}) = \psi_i(\underline{\rho}, \underline{\rho})$, and so it must be that $\varphi_i(\underline{\rho}) = 0$ for each firm i . Since the costs are strictly convex, then it must further be that $s_i(\underline{\rho}) = 0$ for each firm i , else $\varphi_i(\underline{\rho}) > 0$. Again by definition of $\underline{\rho}$, it must be that $D(x) < s_1(x) + s_2(x)$ for all $x > \underline{\rho}$, and since each s_i is upper semicontinuous, then we may conclude that $D(\underline{\rho}) = 0$. This implies that $\varphi_i(x) = 0$ for all prices x , contradicting Assumption 4. ■

Consider a specification of the game with constant marginal cost $\underline{a} \geq 0$. The following proposition shows that in this case, all pure strategy equilibria of Type B must be the classical Bertrand marginal cost pricing $x^* = \underline{a}$ and all Type C equilibria must be market clearing pricing up to capacity as in Cournot $D(x^*) = k_1 + k_2$.

Proposition 5 *If each firm has a constant marginal cost $\underline{a} \geq 0$, then all Type B equilibria are such that $\underline{\rho} = \underline{a}$ and all Type C equilibria are such that $D(\underline{\rho}) = k_1 + k_2$.*

Proof of Proposition 5. Suppose that is $\underline{\rho} > \underline{a}$ and $D(\underline{\rho}) \leq \min k_i$. At $\underline{\rho}$ the front-side profit is $D(\underline{\rho})(\underline{\rho} - \underline{a}) > 0$ and the residual profit is zero, contradicting the definition of $\underline{\rho}$. The second part of the lemma is immediate, since $s_i(p_i) = k_i$ for all $p_i \geq \underline{a}$. ■

Proposition 5 is significant in that it establishes that, regardless of the rationing scheme, all pure strategy equilibria are either classical Bertrand marginal cost pricing or market clearing Cournot pricing in the constant marginal cost setting.

Thus far, our general approach has allowed us to identify when a pure strategy equilibrium exists as well as the character of such an equilibrium. Without explicit reference to the underlying structure of the model, one might infer that pure strategy pricing should emerge in the absence of capacity constraints, provided that costs are sufficiently convex. By adding

standard differentiability assumptions, we are able to show that the existence of a pure strategy equilibrium is remarkably fragile. The following proposition demonstrates that, when the components of the model are smooth, then firms must be capacity constrained for a pure strategy equilibrium to exist.

Proposition 6 *Suppose that c_i, D , and λ_i are differentiable for each firm i . Then in any Type C equilibrium, $s_i(\underline{\rho}) \geq k_i$ for each firm i . Consequently, if costs are strictly convex, then a pure strategy equilibrium cannot exist in the absence of capacity constraints.*

Proof of Proposition 6. Suppose that $s_i(\underline{\rho}) < k_i$ and that a pure strategy equilibrium exists. Then note that $s_i(\underline{\rho})$ is defined by $c'_i(s_i(\underline{\rho})) = \underline{\rho}$. Note that in any Type C equilibrium, $Q_i^R(\underline{\rho}, \underline{\rho}) = s_i(\underline{\rho})$. By Proposition 1, in any Type C equilibrium, it must be that $\underline{\rho}$ is a maximizer of $\psi_i(p_i, \underline{\rho}) = Q_i^R(p_i, \underline{\rho})p_i - c_i(Q_i^R(p_i, \underline{\rho}))$. Differentiating this with respect to p_i yields

$$\frac{\partial}{\partial p_i} \psi_i(p_i, \underline{\rho}) = Q_i^R(p_i, \underline{\rho}) + (p_i - c'_i(Q_i^R(p_i, \underline{\rho}))) \frac{\partial}{\partial p_i} Q_i^R(p_i, \underline{\rho}),$$

where c'_i denotes the derivative of c_i . Using the condition that $Q_i^R(\underline{\rho}, \underline{\rho}) = s_i(\underline{\rho})$, note that

$$\begin{aligned} \frac{\partial}{\partial p_i} \psi_i(\underline{\rho}, \underline{\rho}) &= Q_i^R(\underline{\rho}, \underline{\rho}) + (\underline{\rho} - c'_i(s_i(\underline{\rho}))) \frac{\partial}{\partial p_i} Q_i^R(\underline{\rho}, \underline{\rho}) \\ &= Q_i^R(\underline{\rho}, \underline{\rho}). \end{aligned}$$

Thus, a marginal increase in price will increase the profit of firm i . The second result follows from the fact that $k_i = \infty$ when firms are not capacity constrained and since Proposition 4 guarantees that any equilibrium must be Type C. ■

This proposition demonstrates the fragility of pure strategy equilibria in the BE game. While still possible, they require either a Bertrand equilibrium with (constant) marginal cost pricing or the presence of capacity constraints. By allowing for kinks in the cost functions, as we do in the general case, it is possible to obtain a pure strategy equilibrium without capacity constraints. Note that kinks in the cost function correspond to jumps in the marginal cost of production. If the price is such that the cost function is kinked at a firm's supply, then the firm is unwilling to increase its production any further since the price is less than the marginal cost of additional production. The firm is also unwilling to reduce production since the price exceeds the marginal cost of reducing production.

4 Mixed Strategy Equilibria

Given that pure strategy equilibria will often fail to exist, it is natural to explore the character of the more plausible mixed strategy equilibria of the model. In this section we characterize

the bounds on equilibrium expected profits. Because of the general structure of each firm's residual profit, there can be multiple non-payoff equivalent equilibria and, consequently, we do not provide a precise characterization of equilibrium payoffs. We begin by dealing with the problem of existence of equilibrium.

Proposition 7 *A mixed strategy equilibrium of the BE game exists.*

If the supply functions s_i are continuous, then existence of equilibrium for the BE duopoly follows directly from Proposition 2 in Allison and Lepore (2014). A generalization of this proposition is presented in the Appendix that applies to the case in which the supply functions are not continuous.

Now we state the primary characterization of the mixed strategy equilibria of the BE game, that the expected profits of all equilibria lie between the safe profit and the judo profit. Let u_i^* denote firm i 's equilibrium expected profit.

Proposition 8 *All equilibria are such that $u_i^* \in [\underline{\varphi}_i, \overline{\varphi}_i]$.*

The proof of the Proposition 8 is based on Lemma 1 and the following two lemmas that provide character to all equilibria in order to establish the bounds on equilibrium expected payoffs. The first of these establishes a key property of any equilibrium, that the infimum of the support of each firm's strategy is identical.

Lemma 2 *In any equilibrium, the infimum of the support of each firm's strategy is identical. That is, $\underline{x}_1 = \underline{x}_2 = \underline{x}$. If $\underline{x} > \underline{\rho}$, then neither firm's equilibrium strategy may have an atom at \underline{x} .*

An important implication of Lemma 2 is that in equilibrium each firm selects prices arbitrarily close to \underline{x} with positive probability, receiving its front-side profit with probability arbitrarily close to one. Consequently, each firm i 's expected profit in equilibrium is thus $u_i^* = \varphi_i(\underline{x})$.

Proof of Lemma 2. We begin by proving that in any equilibrium, both firms must choose prices that approach or are equal to \underline{x} with positive probability. That is for both firms i , $M_i(x) < 1$ for all $x > \underline{x}$. Suppose to the contrary that given the equilibrium μ , $\underline{x}_j > \underline{x}_i$ for some firm i . If firm i chooses a price $x < \underline{x}_j$, then $M_i(x) = 1$ and it will receive a payoff of $\varphi_i(x)$. If $\underline{x}_i < \widehat{p}_i$, then by Assumption 1, firm i 's profit is strictly increasing on $[\underline{x}_i, \widehat{p}_i]$. Since \underline{x}_i is in the support of firm i 's equilibrium strategy, it must be that $\underline{x}_i = \widehat{p}_i$ and that $\mu_i(\underline{x}_i) = 1$. As argued in the proof of Proposition 1, firm j must earn a profit of zero. This contradicts Assumption 4, as firm j could choose a price arbitrarily close to \underline{x}_i and guarantee itself a profit arbitrarily close to $\varphi_j(\underline{x}_i) > 0$. We conclude that $\underline{x}_i = \underline{x}_j$.

Next we show that neither firm can have an atom at an $\underline{x} > \underline{\rho}$. Suppose to the contrary that firm j has an atom $\mu_j(\{\underline{x}\}) > 0$ at \underline{x} and we will show a contradiction. From Lemma 1, we know that $\mu_i(\{\underline{x}\}) = 0$, however, as we have just shown, $\underline{x}_i = \underline{x}$. Thus, $\mu_i((\underline{x}, \underline{x} + \delta)) > 0$ for all $\delta > 0$. It is sufficient to show that there is some $x' < \underline{x}$ and neighborhood $(\underline{x}, \underline{x} + \delta)$ such that $\int u_i(x', p_j) d\mu_j > \int u_i(x, p_j) d\mu_j$ for all $x \in (\underline{x}, \underline{x} + \delta)$.

Let $\varepsilon > 0$ be such that $\varepsilon < (1/3)\mu_j(\{\underline{x}\}) (\varphi_i(\underline{x}) - \psi_i(\underline{x}, \underline{x}))$. Since φ_i is continuous on $[0, \widehat{p}_i]$, it must also be uniformly continuous. Let $\delta > 0$ be such that (i) if $|x - x'| < \delta$, then $|\varphi_i(x) - \varphi_i(x')| < \varepsilon$ and (ii) if $|x - \underline{x}| < \delta$, then $|\psi_i(x, \underline{x}) - \psi_i(\underline{x}, \underline{x})| < \varepsilon$. Then consider the price $x' = \underline{x} - (1/4)\delta$ and neighborhood $(\underline{x}, \underline{x} + (1/2)\delta)$. Then note that for all $x \in (\underline{x}, \underline{x} + \delta)$,

$$\begin{aligned} u_i(x', p_j) - u_i(x, p_j) &= \begin{cases} \varphi_i(x') - \psi_i(x, p_j) & \text{if } p_j < x \\ \varphi_i(x') - [\alpha_i(x, x)\varphi_i(x) + (1 - \alpha_i(x, x))\psi_i(x, p_j)] & \text{if } p_j = x \\ \varphi_i(x') - \varphi_i(x) & \text{if } p_j > x \end{cases} \\ &\geq \begin{cases} \varphi_i(x') - \psi_j(x, x) & \text{if } p_j < x \\ \varphi_i(x') - \varphi_i(x) & \text{if } p_j \geq x \end{cases} \\ &= \begin{cases} \varphi_i(x') - \varphi_i(\underline{x}) + \varphi_i(\underline{x}) - \psi_i(\underline{x}, \underline{x}) & \text{if } p_j < x \\ +\psi_i(\underline{x}, \underline{x}) - \psi_i(x, \underline{x}) + \psi_i(x, \underline{x}) - \psi_i(x, x) & \text{if } p_j < x \\ \varphi_i(x') - \varphi_i(x) & \text{if } p_j \geq x \end{cases} \end{aligned}$$

Then, since $\max\{|x' - \underline{x}|, |x' - x|, |x - \underline{x}|\} < \delta$, it follows that $\min\{\varphi_i(x') - \varphi_i(\underline{x}), \varphi_i(x') - \varphi_i(x), \psi_i(\underline{x}, \underline{x}) - \psi_i(x, \underline{x})\} > -\varepsilon$. Moreover, $\psi_i(x, \underline{x}) - \psi_i(x, x) \geq 0$ since ψ_i is weakly decreasing in p_j . Thus, for all $x \in (\underline{x}, \underline{x} + \delta)$,

$$u_i(x', p_j) - u_i(x, p_j) > \begin{cases} \varphi_i(\underline{x}) - \psi_i(\underline{x}, \underline{x}) - 2\varepsilon & \text{if } p_j < x \\ -\varepsilon & \text{if } p_j \geq x \end{cases}.$$

It follows that for all such x ,

$$\int (u_i(x', p_j) - u_i(x, p_j)) d\mu_j > \int_{[\underline{x}, x]} (\varphi_i(\underline{x}) - \psi_i(\underline{x}, \underline{x}) - 2\varepsilon) d\mu_j - \int_{[x, \bar{x}]} \varepsilon d\mu_j.$$

As ε was chosen so that such that $2\varepsilon < \varphi_i(\underline{x}) - \psi_i(\underline{x}, \underline{x})$, it follows that

$$\begin{aligned} \int (u_i(x', p_j) - u_i(x, p_j)) d\mu_j &> \mu_j(\{\underline{x}\}) (\varphi_i(\underline{x}) - \psi_i(\underline{x}, \underline{x}) - 2\varepsilon) - \varepsilon \\ &> \mu_j(\{\underline{x}\}) (\varphi_i(\underline{x}) - \psi_i(\underline{x}, \underline{x})) - 3\varepsilon. \end{aligned}$$

Thus, given our assumption on ε , $\int (u_i(x', p_j) - u_i(x, p_j)) d\mu_j > 0$, and so x' is a profitable deviation from all $x \in (\underline{x}, \underline{x} + \delta)$. Thus, $(\underline{x}, \underline{x} + \delta)$ contains no best responses for firm i , violating μ_i as an equilibrium strategy. We conclude that neither firm may have an atom at \underline{x} if $\underline{x} > \underline{\rho}$. ■

We are now able to prove that the lower bound of any equilibrium must lie between the

critical safe price and the critical judo price.

Lemma 3 *The lower bound of equilibrium pricing \underline{x} is such that $\underline{r} \leq \underline{x} \leq \bar{r}$.*

Proof of Lemma 3. Let μ be an equilibrium. First, we argue that $\underline{x} \leq \bar{r}$. Suppose to the contrary that $\underline{x} > \bar{r}$. This means that $\varphi_i(\underline{x}) > \tilde{\psi}_i(\underline{x})$ for each firm i , and so $\underline{x} > \underline{\rho}$. Without loss of generality, let firm i be such that that \bar{x} is in the support of μ_i and $\mu_j(\{\bar{x}\}) = 0$. Choose $\{x_i^k\}$ in the support of μ_i such that $x_i^k \rightarrow \bar{x}$, then note that

$$\begin{aligned} \lim_{k \rightarrow \infty} \int_{\underline{x}}^{\bar{x}} u_i(x_i^k, p_j) d\mu_j &= \int_{\underline{x}}^{\bar{x}} \psi_i(\bar{x}, p_j) d\mu_j \\ &\leq \int_{\underline{x}}^{\bar{x}} \tilde{\psi}_i(p_j) d\mu_j \\ &\leq \tilde{\psi}_i(\underline{x}) \\ &< \varphi_i(\underline{x}) \\ &= u_i^*. \end{aligned}$$

This contradicts μ_i as an equilibrium strategy. We conclude that $\underline{x} \leq \bar{r}$.

Second, we argue that $\underline{x} \geq \underline{r}$. Suppose to the contrary that $\underline{x} < \underline{r}$. Let player j be such that $\underline{r}_j = \underline{r}$. By definition of \underline{r}_j , it must be that $\underline{u}_i \geq \underline{\varphi}_i(\underline{r})$. Further, since φ_i is strictly increasing on $[\underline{\rho}, \hat{p}_i]$ and $\underline{x} \geq \underline{\rho}$, it follows that $\varphi_i(\underline{x}) < \underline{\varphi}_i(\underline{r})$. The previous lemma implies that $u_i^* = \varphi_i(\underline{x})$. Since firm i can guarantee itself a profit arbitrarily close to \underline{u}_i , this contradicts μ_i as an equilibrium strategy. We conclude that $\underline{x} \geq \underline{r}$. ■

Proof of Proposition 8. It follows from Lemma 2 that each firm i 's equilibrium expected profit is $u_i^* = \varphi_i(\underline{x})$. The statement of the proposition thus follows from Lemma 3 and the facts that φ_i is strictly increasing on $[\underline{\rho}, \hat{p}_i]$ and that $\underline{\rho} \leq \underline{r}$. ■

These payoff bounds provide a solid foundation for understanding the properties of the equilibria of BE games in a general setting. While the literature on BE games has in some cases been able to provide precise payoff predictions, the bounds presented here apply to a much larger class of games than previously studied. The proposition gives precise predictions of the equilibrium profits when the judo price and safe price coincide ($\underline{r} = \bar{r}$). It is worth pointing out that this condition still generalizes previously studied settings with symmetric constant marginal cost and the efficient rationing in which precise predictions are possible.

In order to demonstrate some additional character of equilibrium strategies and their supports, we need to formally define maximizers of the residual profit function conditional on the other firm's mixed strategy. Given a distribution of prices F_j , define the set of *conditional residual maximizers* $\hat{P}_i(F_j) \equiv \arg \max_x E_{F_j}[\psi_i(x, p_j) | p_j \leq x]$. We use $\underline{p}_i(F_j)$ and $\bar{p}_i(F_j)$ to denote the smallest and largest conditional residual maximizer, respectively. That

is, $\underline{p}_i(F_j) = \min \tilde{P}_i(F_j)$ and $\bar{p}_i(F_j) = \sup \tilde{P}_i(F_j)$, where the right continuity of F_j ensures that $\tilde{P}_i(F_j)$ contains a minimal element, while it need not contain a maximal element. The following lemma demonstrate that the upper bound on pricing must lie between the smallest and largest of all firms' conditional residual maximizers.

Lemma 4 *In any equilibrium $F = (F_1, F_2)$, $\min\{\underline{p}_1(F_2), \underline{p}_2(F_1)\} \leq \bar{x} \leq \max\{\bar{p}_1(F_2), \bar{p}_2(F_1)\}$.*

Proof of Lemma 4. Suppose to the contrary that either $\bar{x} < \min\{\underline{p}_1(F_2), \underline{p}_2(F_1)\}$ or $\bar{x} > \max\{\bar{p}_1(F_2), \bar{p}_2(F_1)\}$. If $\bar{x} = \underline{\rho}$ the result is immediately satisfied. Else, from Lemma 1, at most one firm may have an atom at \bar{x} and so there is some firm i with an equilibrium expected profit of

$$\begin{aligned} u_i &= u_i(\bar{x}, F_j) \\ &= E_{F_j}[\psi_i(\bar{x}, p_j) | p_j \leq \bar{x}]. \end{aligned}$$

Thus \bar{x} is a best response for firm i . It follows from our supposition that $\bar{x} \notin \tilde{P}_i(F_j)$. Note that for any price x ,

$$\begin{aligned} u_i(x, F_j) &= M_i(x)\varphi_i(x) + \int_{[\underline{x}, x]} \psi_i(x, p_j) dF_j \\ &\geq (1 - F_j(x))\varphi_i(x) + F_j(x)E_{F_j}[\psi_i(x, p_j) | p_j \leq x]. \end{aligned}$$

By definition of $\tilde{P}_i(F_j)$, $E_{F_j}[\psi_i(x, p_j) | p_j \leq x] > E_{F_j}[\psi_i(\bar{x}, p_j) | p_j \leq \bar{x}]$ for all $x \in \tilde{P}_i(F_j)$. Thus, since $\varphi_i(x) \geq \psi_i(x, p_j)$ for all p_j , then $u_i(x, F_j) > u_i(\bar{x}, F_j)$ for all $x \in \tilde{P}_i(F_j)$. This contradicts \bar{x} as a best response. We conclude that $\min\{\underline{p}_1(F_2), \underline{p}_2(F_1)\} \leq \bar{x} \leq \max\{\bar{p}_1(F_2), \bar{p}_2(F_1)\}$. ■

Remark 4 *In contrast to the seminal work of Edgeworth (1925) and Shubik (1959), the generality of our specification introduces an additional level of pricing indeterminacy. The first level of indeterminacy, which Edgeworth and Shubik focus on, is based on the equilibrium being in non-degenerate mixed strategies. The second level of indeterminacy present in this framework is driven by the fact that there can be multiple non-payoff equivalent equilibria.*

The preceding analysis of this section provides abstract bounds on range of pricing for all equilibria, which contains the range of total indeterminacy. Lemma 3 directly states that the lower bound of all equilibria must lie between the critical safe price and critical judo price. As noted previously, all equilibria will have prices bounded below the maximum of the two firms monopoly price and based on Lemma 4 the least upper bound on pricing must be weakly greater than the minimum of all residual maximizers across the firms. This provides abstract bounds on the range of pricing for all equilibria.

5 Demand and Supply Shifts

As there is no means by which the mixed strategy equilibria of the general model can be directly computed, it is not possible to provide comparative statics directly on such equilibria. Even with full knowledge of the equilibrium strategies, such a task may be impossible if there are multiple equilibria, as the potential for switching between equilibria can disrupt any consistent comparative statics. Instead, we examine the effects of changes in supply, demand, and consumer rationing on the bounds on equilibrium prices and payoffs. It should be clear that, for small changes to these components, the actual equilibrium payoffs need not follow the bounds (though perhaps are likely to). However, for sufficiently large changes, when the new range of prices or profits does not intersect the old, we are able to precisely conclude how the actual equilibrium profits are effected.

We begin by examining the role of changes to the demand side of the market. Specifically, we consider an increase in demand ($D \rightarrow D'$, $D' > D$) or a change in the rationing of consumers, whereby more consumers are rationed ($\lambda \rightarrow \lambda'$, $\lambda' < \lambda$). Given market demands D and D' and rationing rules $\lambda = (\lambda_1, \lambda_2)$ and λ' , let $\varphi = (\varphi_1, \varphi_2)$ and φ' denote the corresponding front-side profits and $\psi = (\psi_1, \psi_2)$ and ψ' denote the corresponding residual profits.

Proposition 9 *Suppose either that demand increases or that rationing becomes more generous, that is, $D' > D$ or $\lambda' < \lambda$ for $p_1 \neq p_2$ and that $\hat{p}'_i \geq \hat{p}_i$. Then the critical judo and safe prices weakly increase, that is, $\bar{r}' \geq \bar{r}$ and $\underline{r}' \geq \underline{r}$. Consequently, the bounds on equilibrium expected profits weakly increase ($\bar{\varphi}'_i \geq \bar{\varphi}_i$, and $\underline{\varphi}'_i \geq \underline{\varphi}_i$).*

To understand the need for weak inequalities as relations on prices and expected profits in the proposition, it is useful to recall that the classical symmetric constant marginal cost Bertrand game fits within our general structure. Since the unique equilibrium in the classical Bertrand game is to price at marginal cost, the equilibrium prices and profit are unaffected by a demand or rationing shift.

Proof of Proposition 9. The result is trivial if $\bar{r} = \underline{\rho}$ since $\underline{\rho}' \geq \underline{\rho}$. Suppose that $\bar{r} > \underline{\rho}$ and without loss of generality assume that $\bar{r}_i = \bar{r}$. Then it must be the case that $D(\bar{r}) > \max\{s_1(\bar{r}), s_2(\bar{r})\}$. Since $D' \geq D$, it follows that $D'(x) > \max\{s_1(x), s_2(x)\}$ for all $x \leq \bar{r}$. Thus, $Q_i(x) = Q'_i(x)$ and so $\varphi_i(x) = \varphi'_i(x)$ for all $x \leq \bar{r}$. Since $\lambda' \leq \lambda$, then $\tilde{\psi}'_i(x) \geq \tilde{\psi}_i(x)$ for all $x \leq \bar{r}$. Thus, for all $x \leq \bar{r}$, if $\psi_i(x) \geq \varphi_i(x)$, then $\tilde{\psi}'_i(x) \geq \varphi'_i(x)$, and since $\psi_i(x) \geq \varphi_i(x)$ for all such x by the definition of \bar{r} , it follows immediately that

$$\left\{ p_i \in [\underline{\rho}, \hat{p}_i] \mid \varphi_j(p_i) \leq \tilde{\psi}_j(p_i) \right\} \subset \left\{ p_i \in [\underline{\rho}, \max\{\hat{p}_i, \hat{p}'_i\}] \mid \varphi_j(p_i) \leq \tilde{\psi}'_j(p_i) \right\}.$$

This implies that

$$\sup \left\{ p_i \in [\underline{\rho}, \max\{\hat{p}_i, \hat{p}'_i\}] \mid \varphi_j(p_i) \leq \tilde{\psi}'_j(p_i) \right\} \geq \bar{r}_i.$$

From the assumption that $\widehat{p}'_i \geq \widehat{p}_i$, we conclude that $\bar{r}'_i \geq \bar{r}_i$.

Next, note that $Q'_j(p_j) > Q_j(p_j)$ if and only if $Q_j(p_j) = D(p_j)$. Thus, $D'(p_i) - \lambda'_j(p)Q'_j(p_j) \geq D(p_i) - \lambda_j(p)Q_j(p_j)$, and since $Q_i^{r'}(p) = \min\{D'(p_i) - \lambda'_j(p)Q'_j(p_j), s_i(p_i)\}$, then $Q_i^{r'} \geq Q_i^r$. Thus, given our assumptions on the profit function $\pi_i(p_i, q_i)$, it follows that $\psi'_i \geq \psi_i$. Therefore, since $\underline{u}_i = \max_x \psi_i(x, x)$, it follows immediately that $\underline{u}'_i \geq \underline{u}_i$. Repeating the steps from the first part of this proof, we obtain

$$\{p_i \in [\underline{\rho}, \widehat{p}_i] \mid \varphi_j(p_i) \leq \underline{u}_j\} \subset \{p_i \in [\underline{\rho}, \max\{\widehat{p}_i, \widehat{p}'_i\}] \mid \varphi_j(p_i) \leq \underline{u}'_j\},$$

which implies that

$$\sup \{p_i \in [\underline{\rho}, \min\{\widehat{p}_i, \widehat{p}'_i\}] \mid \varphi_j(p_i) \leq \underline{u}'_j\} \geq \underline{r}_i.$$

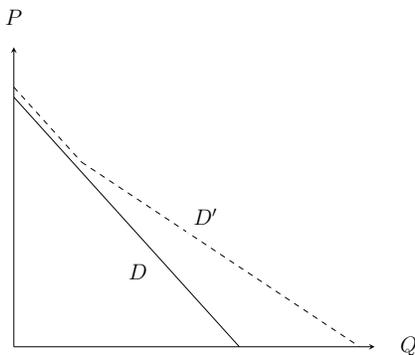
The conclusion follows from the fact that $\widehat{p}'_i \geq \widehat{p}_i$. ■

Note that Proposition 9 only provides a loose characterization on the lower bound of equilibrium pricing. This does not inform the full distribution of equilibrium prices or even the upper bound on pricing. The reason is that an increase in demand or similar change in rationing does not necessarily result in a monotonic shift in the equilibrium pricing distributions. Indeed, the following example demonstrates that the upper bound on equilibrium pricing may decrease after an increase in demand.

Consider a symmetric duopoly with constant marginal cost of production zero, capacities $k \in (4, 12)$, market demand $D(x) = 12 - x$, and efficient rationing. It is straightforward to verify that the upper bound of the equilibrium pricing is given by $\bar{x} = (12 - k)/2$. Consider a demand shift to

$$D'(x) = \begin{cases} 12 - x + \varepsilon & \text{if } x > \bar{x} \\ 12 - \bar{x} - (2 + \varepsilon)(x - \bar{x}) + \varepsilon & \text{if } x \leq \bar{x} \end{cases},$$

as depicted below.



It is again easy to verify that the new upper bound on equilibrium prices is given by

$$\bar{x}' = \frac{12 - k + \varepsilon + \frac{12-k}{2}\varepsilon}{2(1 + \varepsilon)},$$

and it can be shown that $\bar{x} > \bar{x}'$.

This example demonstrates that a non-parallel shift in demand may warp the incentives of the firms so that the upper bound on pricing shifts in a different direction than the lower bound. In this case, the market effectively becomes more elastic, inducing a reduction in the highest prices set by firms in equilibrium. Such a shift does not necessarily have this impact on the upper bound of pricing, indeed if the demand were sufficiently elastic prior to the shift, the upper bound would still increase. This highlights the impracticality of generally characterizing the impact of arbitrary demand and rationing shifts on the whole distribution of pricing.

Now we turn our attention to understanding the impact of changes in production technology. We first show that an increase in a firm's supply, which could result from an increase in capacity or reduction in costs, will reduce its judo and safe prices and (weakly) reduce the profit of the other firm. We then demonstrate via example that a firm's profits may decrease with either a decrease in industry costs or in only its own cost.

Given supply functions $s = (s_1, s_2)$ and s' , let $\varphi = (\varphi_1, \varphi_2)$ and φ' denote the corresponding front-side profits and $\psi = (\psi_1, \psi_2)$ and ψ' denote the corresponding residual profits.

Proposition 10 *Suppose that firm i 's supply increases so that $s'_i > s_i$ for all prices $x > \underline{\rho}$, where the new cost function c'_i is such that $c'_i(q) - c_i(q)$ is weakly increasing. Then critical judo and safe prices weakly decrease, so that $\bar{r}' \leq \bar{r}$ and $\underline{r}' \leq \underline{r}$. Consequently, the bounds on firm j 's equilibrium expected profits weakly decrease ($\bar{\varphi}'_j \leq \bar{\varphi}_j$, and $\underline{\varphi}'_j \leq \underline{\varphi}_j$).*

Proof of Proposition 10. Suppose that $s'_i > s_i$. Recall that firm i 's judo and safe prices are defined by

$$\begin{aligned} \bar{r}_i &= \sup \left\{ p_i \in [\underline{\rho}, \hat{p}_i] \mid \varphi_j(p_i) \leq \tilde{\psi}_j(p_i) \right\} \text{ and} \\ \underline{r}_i &= \sup \left\{ p_i \in [\underline{\rho}, \hat{p}_i] \mid \varphi_j(p_i) \leq \underline{u}_j \right\}. \end{aligned}$$

Note that $\varphi'_j(p_i) = \varphi_j(p_i)$ for all prices since firm j 's front-side profit does not depend on firm i 's supply. However, since firm j 's residual profit is decreasing in the quantity produced by firm i , then $\psi_j(p_i, p_j) \geq \psi'_j(p_i, p_j)$ for all $p_i \leq p_j$. Consequently, $\tilde{\psi}_j(p_i) \geq \tilde{\psi}'_j(p_i)$ and $\underline{u}_j \geq \underline{u}'_j$ since $\underline{u}_j = \max_{x \in [\underline{\rho}, \hat{p}_j]} \psi_j(x, x)$. It follows from the fact that φ_j is strictly increasing that

$$\begin{aligned} \left\{ p_i \in [\underline{\rho}, \min\{\hat{p}_i, \tilde{p}'_i\}] \mid \varphi_j(p_i) \leq \tilde{\psi}'_j(p_i) \right\} &\subset \left\{ p_i \in [\underline{\rho}, \hat{p}_i] \mid \varphi_j(p_i) \leq \tilde{\psi}_j(p_i) \right\} \text{ and} \\ \left\{ p_i \in [\underline{\rho}, \min\{\hat{p}_i, \hat{p}'_i\}] \mid \varphi_j(p_i) \leq \underline{u}'_j \right\} &\subset \left\{ p_i \in [\underline{\rho}, \hat{p}_i] \mid \varphi_j(p_i) \leq \underline{u}_j \right\}. \end{aligned}$$

This implies that

$$\begin{aligned} \sup \left\{ p_i \in [\underline{\rho}, \min\{\widehat{p}_i, \widehat{p}'_i\}] \mid \varphi_j(p_i) \leq \widetilde{\psi}'_j(p_i) \right\} &< \bar{r}_i \text{ and} \\ \sup \left\{ p_i \in [\underline{\rho}, \min\{\widehat{p}_i, \widehat{p}'_i\}] \mid \varphi_j(p_i) \leq \underline{u}'_j \right\} &< \underline{r}_i. \end{aligned}$$

If $\widehat{p}'_i \leq \widehat{p}_i$, then $\bar{r}'_i < \bar{r}_i$ and $\underline{r}'_i < \underline{r}_i$. Otherwise, it is sufficient to show that $\bar{r}'_i \leq \widehat{p}_i$. Suppose that $\widehat{p}'_i > \widehat{p}_i$. Note that $D(\widehat{p}_i) \leq s_i(\widehat{p}_i)$, and so it must be that $D(\widehat{p}'_i) \leq s'_i(\widehat{p}_i)$. It therefore follows that $\widetilde{\psi}_j(\widehat{p}_i) = \widetilde{\psi}'_j(\widehat{p}_i) = 0 < \varphi_j(\widehat{p}_i)$, and thus, $\max\{\bar{r}_i, \bar{r}'_i\} \leq \widehat{p}_i$. We conclude that $\bar{r}'_i \leq \bar{r}_i$ and $\underline{r}'_i \leq \underline{r}_i$.

Next consider firm j 's judo price. Let $\tilde{p} \in \tilde{P}(\bar{r}_j)$ and $\tilde{p}' \in \tilde{P}(\bar{r}'_j)$. To demonstrate that

$$\left\{ p_j \in [\underline{\rho}, \widehat{p}_j] \mid \varphi'_i(p_j) \leq \widetilde{\psi}'_i(p_j) \right\} \subset \left\{ p_j \in [\underline{\rho}, \widehat{p}_j] \mid \varphi_i(p_j) \leq \widetilde{\psi}_i(p_j) \right\},$$

it will be sufficient to show that $\widetilde{\psi}'_i(p_j) - \varphi'_i(p_j) \leq \widetilde{\psi}_i(p_j) - \varphi_i(p_j)$, or equivalently, that $\widetilde{\psi}'_i(p_j) - \widetilde{\psi}_i(p_j) \leq \varphi'_i(p_j) - \varphi_i(p_j)$.

We will make use of the fact that $Q_i(\bar{r}_j) \geq Q_i^r(\tilde{p}, \bar{r}_j)$. To see this, note that the continuity of φ_i and ψ_i in p_i and the lower semicontinuity of ψ_i in p_j imply that, by definition of \bar{r}_j , $\varphi_i(\bar{r}_j) \geq \psi_i(\tilde{p}, \bar{r}_j)$. If $Q_i(\bar{r}_j) < Q_i^r(\tilde{p}, \bar{r}_j)$, then $\pi_i(\tilde{p}, Q_i^r(\tilde{p}, \bar{r}_j)) \geq \pi_i(\tilde{p}, Q_i(\bar{r}_j)) > \pi_i(\bar{r}_j, Q_i(\bar{r}_j))$ since $Q_i^r(\tilde{p}, \bar{r}_j) \leq s_i(\tilde{p})$ and $\tilde{p} > \bar{r}_j$. Thus, it must be that $Q_i(\bar{r}_j) \geq Q_i^r(\tilde{p}, \bar{r}_j)$.

For notational convenience, in the remainder of the proof let $q = Q_i(\bar{r}_j)$, $\tilde{q} = Q_i^r(\tilde{p}, \bar{r}_j)$, $q' = Q_i(\bar{r}'_j)$, and $\tilde{q}' = Q_i^r(\tilde{p}', \bar{r}'_j)$. Since c_i and c'_i are convex, they are differentiable almost everywhere and there exist functions $\chi > \chi'$ such that $c_i(Q) = \int_{[0, Q]} \chi(z) dz$ and $c'_i(Q) = \int_{[0, Q]} \chi'(z) dz$. With this notation, note that

$$\begin{aligned} \varphi'_i(p_j) - \varphi_i(p_j) &= \pi'_i(p_j, q') - \pi_i(p_j, q) \\ &= \pi'_i(p_j, q') - \pi'_i(p_j, q) + \pi'_i(p_j, q) - \pi_i(p_j, q) \\ &\geq \pi'_i(p_j, q) - \pi_i(p_j, q) \\ &= c_i(q) - c'_i(q) \\ &= \int_{[0, q]} (\chi(z) - \chi'(z)) dz \end{aligned}$$

and

$$\begin{aligned}
\tilde{\psi}'_i(p_j) - \tilde{\psi}_i(p_j) &= \psi'_i(\tilde{p}', p_j) - \psi_i(\tilde{p}', p_j) \\
&\leq \psi'_i(\tilde{p}', p_j) - \psi_i(\tilde{p}', p_j) \\
&= \pi'_i(\tilde{p}', \tilde{q}') - \pi_i(\tilde{p}', \tilde{q}') \\
&\leq \pi'_i(\tilde{p}', \tilde{q}') - \pi_i(\tilde{p}', \tilde{q}') \\
&= c_i(\tilde{q}') - c'_i(\tilde{q}') \\
&= \int_{[0, \tilde{q}']} (\chi(z) - \chi'(z)) dz.
\end{aligned}$$

Thus, since $q \geq \tilde{q}$, $\varphi'_i(p_j) - \varphi_i(p_j) \geq \int_{[0, q]} (\chi(z) - \chi'(z)) dz \geq \int_{[0, \tilde{q}']} (\chi(z) - \chi'(z)) dz \geq \tilde{\psi}'_i(p_j) - \tilde{\psi}_i(p_j)$. We may thus conclude that

$$\left\{ p_j \in [\underline{\rho}, \hat{p}_j] \mid \varphi'_i(p_j) \leq \tilde{\psi}'_i(p_j) \right\} \subset \left\{ p_j \in [\underline{\rho}, \hat{p}_j] \mid \varphi_i(p_j) \leq \tilde{\psi}_i(p_j) \right\},$$

and consequently that $\bar{r}'_j \leq \bar{r}_j$. Therefore $\bar{r}' \leq \bar{r}$.

Finally, consider firm j 's safe price. Suppose to the contrary that $\underline{r}'_j > \underline{r}_j$. By construction, $\varphi_i(\underline{r}_j) = \underline{u}_i$. Thus for all $x > \underline{r}_j$, $Q_i^r(x, x) < Q_i(\underline{r}_j)$, else $\psi_i(x, x) > \varphi_i(\underline{r}_j)$. It follows that $s_i(x) > Q_i^r(x, x)$ for all $x > \underline{r}_j$, and further, that $s'_i(x) > Q_i^{r'}$ for all $x > \underline{r}_j$. This implies that $Q_i^r(x, x) = Q_i^{r'}(x, x)$ for all $x > \underline{r}_j$. Thus, $\psi'_i(x, x) = \psi_i(x, x) + c_i(Q_i^r(x, x)) - c'_i(Q_i(x, x))$ for all $x > \underline{r}_j$.

Let x^n be a sequence of prices such that $\psi'(x^n, x^n) \rightarrow \underline{u}'_i$. Without loss of generality, we may choose this sequence so that $\psi_i(x^n, x^n)$ converges. Note that for all n ,

$$\begin{aligned}
\varphi'_i(\underline{r}'_j) - \varphi_i(\underline{r}'_j) &\geq c_i(Q_i(\underline{r}_j)) - c'_i(Q_i(\underline{r}_j)) \\
&\geq c_i(Q_i^r(x^n, x^n)) - c'_i(Q_i^r(x^n, x^n)) \\
&= \psi'_i(x^n, x^n) - \psi_i(x^n, x^n).
\end{aligned}$$

Thus, $\varphi'_i(\underline{r}'_j) - \varphi_i(\underline{r}'_j) \geq \underline{u}'_i - \lim_n \psi_i(x^n, x^n) \geq \underline{u}'_i - \underline{u}_i$. We may conclude that $\varphi'_i(\underline{r}'_j) - \underline{u}'_i \geq \varphi_i(\underline{r}'_j) - \underline{u}_i$. However, since $\underline{r}'_j > \underline{r}_j$, then by definition of \underline{r}_j , $\varphi_i(\underline{r}'_j) > \underline{u}_i$. It follows that $\varphi'_i(\underline{r}'_j) > \underline{u}'_i$, contradicting \underline{r}'_j as firm j 's judo price. Consequently, we conclude that $\underline{r}'_j \leq \underline{r}_j$.

The final statement of the proposition follows from the fact that a decrease in firm i 's judo and safe prices weakly reduces the critical judo and safe prices and that the front-side profit is strictly increasing. ■

Note that this proposition does not make any statements regarding the profits of the firm whose supply shifts. The reason is that the effect is ambiguous. That is, a technology increase for a firm does not necessarily imply an increase in equilibrium profits for that firm. The direction of the change in profits is instead determined by the nature of the shift as well as market conditions. Two extreme examples illustrate this point.

Consider a duopoly in which identical firms have constant marginal cost and capacities equal to half the monopoly quantity. In such a setting, pure strategy pricing can be sustained with each firm earning half the monopoly profit. Now consider a technology shock that increases the capacity of both firms so that their capacity is nonbinding at any price. This technology increase actually lowers each firm's profit from something strictly positive to zero.

The previous example involved an industry-wide capacity shock, however, the same result may occur as a result of a cost reduction for a single firm. Consider a duopoly in which firm 1 has constant marginal cost $c = 0$ while firm 2 has a strictly convex cost of production with supply $s_2(x) > 0$ for all $x > 0$ and $s_2(0) = 0$. Suppose that the firms are not capacity constrained. It follows that $\underline{\rho} = 0$. Note that by choosing a price x arbitrarily close to zero, the right continuity of s_2 guarantees that $\psi_1(x, 0) > 0$. Thus, $p_1 = p_2 = 0$ cannot be an equilibrium and any equilibrium must be in mixed strategies. Therefore, it must be that firm 2 receives its front end profit in equilibrium with positive probability, in which case it earns positive profits. Consider a technology increase of firm 2 that reduces its cost to zero. Then the game becomes the classic Bertrand duopoly with zero profits. Thus, a reduction in one firm's cost may actually reduce its profits.

The key insight that these examples highlight is that there are countervailing effects associated with a change in technology. There is a primary cost effect or capacity effect that allows a firm to earn a higher profit margin or produce more at any given price, both of which increase the profits of that firm. Alternatively, there is a secondary competition effect, whereby the change in cost or capacity alters the strategic environment and incentivizes the other firm to price more competitively, driving the prices of both firms down and thereby reducing profits. Whether the net change in profits is positive or negative depends on the relative strength of these two effects.

6 Special Cases with Unique Equilibria

In this section we examine two special cases of the general model. We first consider a case with symmetric firms with strong concavity assumptions. The concavity assumptions are useful to eliminate the possibility of gaps in the support of mixed strategy equilibria. We find it interesting that concavity enables the proof of equilibrium uniqueness in this mixed strategy setting through an entirely different channel than it does in the canonical argument for unique pure strategy equilibrium in smooth games. Second, we consider a case in which the residual profits are independent of the lower price. In this case, we are able to derive a closed form solution for the unique equilibrium.

6.1 Symmetry and Uniqueness of Equilibrium

We demonstrate that there is a unique Nash equilibrium when firms are symmetric, the front-side profits are strictly concave, and the residual profits are weakly concave. Even under these restrictions, significant analysis is required to demonstrate uniqueness, and we are able to show that this technique does not generalize to the analysis of asymmetric firms. Nevertheless, this analysis does provide some insight into why it is not possible to abstractly derive a single expected payoff or provide comparative statics on the actual equilibrium payoffs. In particular, we are able to identify the possibility for multiple “types” of non-payoff equivalent equilibria which may simultaneously exist when firms are not symmetric.

It is worthwhile to point out that the technical results of this section apply to the case in which firms are not identical. Only the final uniqueness results requires that firms be symmetric.

The assumptions we require are as follows.

Assumption 5 φ_i is strictly concave on $[\underline{\rho}, \widehat{p}_i]$.

For any price x , let $\omega_i(x) = \sup\{p_i : \psi_i(p_i, x) > 0\}$ denote the lowest price at which firm i 's residual profit is zero. Further define $\varpi_i = \sup\{x \leq x_i^m : \psi_i(x, x) > 0\}$.

Assumption 6 $\psi_i(p_i, p_j)$ is concave in p_i on $[p_j, \omega_i(p_j))$.

The following proposition implies that in any equilibrium, each firm must mix in such a way that it might select any price between the lowest price set \underline{x} and the lesser of the two firms' highest price, $\min\{\bar{x}_1, \bar{x}_2\}$. Furthermore, neither firm's strategy may have an atom at any price below $\min\{\bar{x}_1, \bar{x}_2\}$.

Proposition 11 *Given Assumptions 5 and 6, in any equilibrium $F = (F_1, F_2)$, either each F_i is continuous and strictly increasing on $[\underline{x}, \min\{\bar{x}_1, \bar{x}_2\})$ or there exist prices a and b such that $a < \min\{\varpi_1, \varpi_2\} \leq b$ such that each F_i is continuous and strictly increasing on $[\underline{x}, a)$ and constant on $[a, b)$.*

The proof of this proposition will require the following two technical lemmas. The proofs of these lemmas are located in the Appendix.

Lemma 5 *Given Assumptions 5 and 6, in any equilibrium $F = (F_1, F_2)$, if F_i is constant on some interval $[a, b) \subset [\underline{x}, \min\{\bar{x}_1, \bar{x}_2, \varpi_1, \varpi_2\})$, then F_j is also constant on some interval $[a, b') \subset [\underline{x}, \min\{\bar{x}_1, \bar{x}_2, \varpi_1, \varpi_2\})$.*

Lemma 6 *Given Assumptions 5 and 6, in any equilibrium $F = (F_1, F_2)$, if F_1 and F_2 are constant on some interval $[a, b)$, where $b < \min\{\bar{x}_1, \bar{x}_2, \varpi_1, \varpi_2\}$, a is in the support of F_i , and b is in the support of F_j for some j (possibly $j = i$), then*

- 1) $u_i(x, F_j)$ is strictly decreasing on (a, b) for some firm i , and
- 2) $\mu_j(\{b\}) > 0$ and $u_j(x, F_i)$ is strictly increasing on (a, b) for the firm $j \neq i$.

Proof of Proposition 11. We will first show that if F_i is constant on any interval $[a, b)$ with $a < \min\{\bar{x}_1, \bar{x}_2, \varpi_1, \varpi_2\}$, then $b \geq \min\{\varpi_1, \varpi_2\}$. This will imply that there exists an $a \leq \min\{\bar{x}_1, \bar{x}_2, \varpi_1, \varpi_2\}$ such that F_i is strictly increasing on and $[\underline{x}, a)$ and constant on $[a, \min\{\bar{x}_1, \bar{x}_2, \varpi_1, \varpi_2\})$. Second, we will show that each F_i is continuous on $[\underline{x}, a)$.

Suppose that F_i is constant on some interval $[a, b)$ with $a < [\underline{x}, \min\{\bar{x}_1, \bar{x}_2, \varpi_1, \varpi_2\})$. From Lemma 5, we may choose the interval $[a, b)$ so that F is constant on $[a, b)$ and both a and b are in the union of the supports of F_1 and F_2 . Suppose to the contrary that $b < \min\{\varpi_1, \varpi_2\}$. Based on Lemma 6, there is some firm j such that $\mu_j(\{b\}) > 0$ and u_i is strictly decreasing on $[a, b)$. From Lemma 6, b cannot be in the support of F_i , so there is some $b' > b$ such that F_i is constant on $[a, b')$. Choose b' to be in the union of the supports of F_1 and F_2 . If $b = \bar{x}_j$, then b' is in the support of F_i . Otherwise, since $\mu_j(\{b\}) > 0$, then Lemma 6 implies that $u_j(x, F_i)$ is strictly decreasing on $[b, b')$, and so $\mu_i(\{b'\}) > 0$. We will show the contradiction that $u_i(a, F_j) > u_i(b', F_j)$.

Define for all $x \in (a, b']$ the function

$$\bar{u}_i(x, F_j) = (1 - F_j(a))\varphi_i(x) + \int_{[\underline{x}, a]} \psi_i(x, p_j) d\mu_j,$$

and let $\bar{u}_i(a, F_j) = \lim_{x \rightarrow a^-} u_i(x, F_j)$. Define $\tilde{u}_i(x, F_j)$ be the continuous extension of $u_i(x, F_j)$ from (b, b') to $[b, b']$. Note that \bar{u}_i is defined as if F_j were constant on $[a, b')$, and thus Assumptions 5 and 6 imply that \bar{u}_i is strictly concave on (a, b') . It follows that \bar{u}_i is strictly decreasing on $[a, b']$.

Note that

$$\begin{aligned} \tilde{u}_i(b', F_j) &= (1 - F_j(b))\varphi_i(b') + \int_{[\underline{x}, b]} \psi_i(b', p_j) d\mu_j \\ &= (1 - F_j(a))\varphi_i(b') + \int_{[\underline{x}, a]} \psi_i(b', p_j) d\mu_j \\ &\quad - \mu_j(\{b\})\varphi_i(b') + \mu_j(\{b\})\psi_i(b', b) \\ &= \bar{u}_i(b', F_j) - \mu_j(\{b\})(\varphi_i(b') - \psi_i(b', b)). \end{aligned}$$

Since $\bar{u}_i(x, F_j)$ is strictly increasing on $[a, b']$, then $\bar{u}_i(a, F_j) > \bar{u}_i(b', F_j)$. Further, since $\mu_j(\{b\})(\varphi_i(b') - \psi_i(b', b)) \geq 0$, then $\bar{u}_i(a, F_j) > \tilde{u}_i(b', F_j) = u_i(b', F_j)$. This contradicts

b' as an equilibrium price for firm i . We conclude that each F_i is strictly increasing on $[\underline{x}, \min\{\bar{x}_1, \bar{x}_2\})$.

Let $a \leq \min\{\bar{x}_1, \bar{x}_2, \varpi_1, \varpi_2\}$ be such that F_i is strictly increasing on and $[\underline{x}, a)$ and constant on $[a, \min\{\bar{x}_1, \bar{x}_2, \varpi_1, \varpi_2\})$. We will now show that each F_i is continuous on $[\underline{x}, a)$. Suppose to the contrary that there is some price $x \in (\underline{x}, a)$ such that $\mu_j(\{x\}) > 0$ for some firm j .¹⁶ We will show that there is some interval $[x, x + \delta)$ on which F_i is constant. Note that

$$\lim_{y \rightarrow x^-} u_i(y, F_j) - \lim_{y \rightarrow x^+} u_i(y, F_j) = \mu_j(\{a\})\varphi_i(x) - \mu_j(\{x\})\psi_i(x, x).$$

Since $x > \underline{x} \geq \rho$, then $\varphi_i(x) > \psi_i(x, x)$. Thus, there is some $\delta > 0$ such that $u_i(x - \delta, F_j) > u_i(y, F_j)$ for all $y \in (x, x + \delta)$. It follows that $(x, x + \delta)$ contains no best responses for firm i , and so it must be that F_i is constant on $[x, x + \delta)$. This contradicts the previous part of this proof. We conclude that F_1 and F_2 are continuous on $[\underline{x}, \min\{\bar{x}_1, \bar{x}_2\})$. ■

Proposition 11 identifies two types of equilibrium strategies: one in which firms mix in such a way that they may choose any price between the lower bound \underline{x} and the lesser of the two firms' upper bounds $\min\{\bar{x}_1, \bar{x}_2\}$, and another in which the firms have a gap in their support such that if both firms choose prices above the gap, the firm with the higher price will obtain zero profit. We are unable to rule out the possibility that both types of equilibria can simultaneously exist, owing largely to the fact that each firm's payoffs are very poorly behaved at prices above ϖ_i . Furthermore, these two types of equilibria may have different lower bounds on equilibrium pricing, and thus yield different expected payoffs.

The following proposition demonstrates that when firms are identical, we are able to rule out the type of equilibrium with a gap.

Proposition 12 *Suppose that firms are identical. Given Assumptions 5 and 6, then all equilibria are payoff equivalent. Moreover, the equilibrium is uniquely determined on $[\underline{x}, \min\{\bar{x}_1, \bar{x}_2\})$ and is symmetric and atomless on $[\underline{x}, \min\{\bar{x}_1, \bar{x}_2\})$.*

We will use the following technical result in the proof of Proposition 12. The proof of this lemma is located in the Appendix.

Lemma 7 *Let $f \geq 0$ be nonincreasing on an interval $[a, b]$ and let F and G be distribution functions of probability measures over $[a, b]$. If $F \leq G$ for all $x \in [a, b]$, then $\int f dF \leq \int f dG$.*

Now we complete the proof of uniqueness of equilibrium.

Proof of Proposition 12. Note that $\varpi_1 = \varpi_2 = \varpi$ since firms are identical. We will begin by showing that all equilibria are symmetric on $[\underline{x}, \min\{\bar{x}_1, \bar{x}_2, \varpi\})$. We then use this

¹⁶Since distribution functions are inherently right continuous, each is continuous at \underline{x} .

fact to show that all equilibria are symmetric and atomless on $[\underline{x}, \min\{\bar{x}_1, \bar{x}_2\}]$. We will then argue that given two equilibria with the same lower bound \underline{x} , both must be identical on $[\underline{x}, \min\{\bar{x}_1, \bar{x}_2, \varpi\}]$. Lastly, we will demonstrate that all equilibria are payoff equivalent, which will imply that all equilibria have the same lower bound \underline{x} .

Since the firms are identical, we will drop the subscripts on the front-side and residual profit functions.

Let F be an equilibrium and suppose to the contrary that $F_1 \neq F_2$ on $[\underline{x}, \min\{\bar{x}_1, \bar{x}_2, \varpi_1, \varpi_2\}]$. From Proposition 11, let $a \leq \min\{\bar{x}_1, \bar{x}_2, \varpi\}$ be such that each F_i is continuous and strictly increasing on $[\underline{x}, a]$ and constant on $[a, \min\{\bar{x}_1, \bar{x}_2, \varpi\}]$. Choose $\xi \in (a, \varpi)$ such that $|F_1(x) - F_2(x)| < |F_1(\xi) - F_2(\xi)|$ for all $x < \xi$.¹⁷ Without loss of generality assume that $F_1(\xi) > F_2(\xi)$.

Define for each $x \leq \xi$ the functions $G_2(x) = \min\{F_1(x), F_2(x)\}$ and $G_1(x) = F_1(\xi) - F_2(\xi) + G_2(x)$. Then note that $G_1(x) \geq F_1(x)$ and $G_2(x) \leq F_2(x)$ for all $x \leq \xi$ and $F_i(\xi) = G_i(\xi)$ for each $i = 1, 2$. Thus, from Lemma 7, we have that

$$(1) \quad \frac{1}{F_1(x)} \int_{[\underline{x}, x]} \psi(x, z) dF_1(z) \leq \frac{1}{G_1(x)} \int_{[\underline{x}, x]} \psi(x, z) dG_1(z),$$

$$(2) \quad \frac{1}{F_2(x)} \int_{[\underline{x}, x]} \psi(x, z) dF_2(z) \geq \frac{1}{G_2(x)} \int_{[\underline{x}, x]} \psi(x, z) dG_2(z).$$

Since the firms are identical, then their expected profits must be the same. Thus, for all $x \in [\underline{x}, \xi]$,

$$(1 - F_2(x))\varphi(x) + \int \psi(x, p_2) dF_2 = (1 - F_1(x))\varphi(x) + \int \psi(x, p_1) dF_1,$$

or equivalently,

$$(3) \quad (F_1(x) - F_2(x))\varphi(x) = \int \psi(x, p_1) dF_1 - \int \psi(x, p_2) dF_2.$$

Note that from (1) and (2),

$$\int \psi(\xi, p_1) dF_1 - \int \psi(\xi, p_2) dF_2 \leq \int \psi(\xi, p_1) dG_1 - \int \psi(\xi, p_2) dG_2,$$

and since

$$\int \psi(x, p_1) dG_1 = \int \psi(x, p_2) dG_2 + (F_1(\xi) - F_2(\xi))\psi(x, \underline{x}),$$

¹⁷The existence of such a price is guaranteed by the fact that $F_1(\underline{x}) = F_2(\underline{x})$ and that $F_1 - F_2$ must achieve a maximum on any compact interval $[\underline{x}, x] \subset [\underline{x}, a]$.

then

$$(4) \quad \int \psi(\xi, p_1) dF_1 - \int \psi(\xi, p_2) dF_2 \leq (F_1(\xi) - F_2(\xi))\psi(\xi, \underline{x}).$$

Evaluating (3) at $x = \xi$ and substituting (4) yields

$$\begin{aligned} (F_1(\xi) - F_2(\xi))\varphi(\xi) &\leq (F_1(\xi) - F_2(\xi))\psi(\xi, \underline{x}) \\ (F_1(\xi) - F_2(\xi))(\varphi(\xi) - \psi(\xi, \underline{x})) &\leq 0 \end{aligned}$$

Since $\xi > \underline{\rho}$, then it must be that $\varphi(\xi) - \psi(\xi, \underline{x}) > 0$. It follows that $F_1(\xi) = F_2(\xi)$, a contradiction. Thus, it must be that $F_1 = F_2$ on $[\underline{x}, \min\{\bar{x}_1, \bar{x}_2, \varpi\})$.

Suppose now that $F_1 = F_2$ on $[\underline{x}, \min\{\bar{x}_1, \bar{x}_2, \varpi\})$ and that $F_1(x) \neq F_2(x)$ for some $x \geq \min\{\bar{x}_1, \bar{x}_2, \varpi\}$. Then from Proposition 11, $F_1(x) = F_2(x)$ for all $x < \varpi$. Furthermore, by definition of ϖ , for each firm i and all $x \geq \varpi$ with $\mu_j(\{x\}) = 0$

$$u_i(x, F_j) = (1 - F_j(x))\varphi(x) + \int_{[\underline{x}, \varpi)} \psi(x, z) dF_j(z).$$

Thus, for any $x \geq \varpi$ in the support of F_j , it must be that $F_i(x) = F_j(x)$. It follows that the equilibrium must be symmetric and atomless on $[\underline{x}, \min\{\bar{x}_1, \bar{x}_2\}]$.

To observe that the equilibrium is uniquely determined given the lower bound \underline{x} , note that the previous part of the proof remains true if F_1 and F_2 are taken to be two different equilibrium strategies for the same firm.

Next we show that all equilibria are payoff equivalent. Suppose to the contrary that there are two equilibria F and F' with lower bounds $\underline{x} > \underline{x}'$. Let π_i and π'_i denote firm i 's expected profits in these equilibria. Note that any price x at which a firm j has supply $s_j(x) = 0$ must be such that $x \leq \underline{\rho}$. Since $\underline{x} \geq \underline{\rho}$ in equilibrium, then it must be that $s_j(x) > 0$ for all $x > \underline{x}$. It follows that if $\underline{x} > \underline{x}'$, then $s_j(x) > 0$ for any $x \in (\underline{x}', \underline{x})$. This implies that $\varphi_j(\underline{x}) > 0$, and so $\pi_j > 0$ for each firm j . It follows immediately that $\pi_i > \pi'_i$ for each firm i .

Note that since both $\underline{x} \geq \underline{\rho}$ and $\underline{x}' \geq \underline{\rho}$, then $\underline{\rho} < \underline{x}$. This implies that $F_i(\underline{x}) = 0$ for each player i , else a deviation to $\underline{x} - \varepsilon$ is a profitable deviation for the firm without the atom at \underline{x} for sufficiently small ε .

Since $\underline{x}' < \underline{x}$, then each $F'_i(\underline{x}) > 0$, while $F_i(\underline{x}) = 0$. Define $y_i > \underline{x}$ to be the lowest price such that $F_i(y_i) \geq F'_i(y_i)$, that is, $y_i = \sup\{x : F_i(x) < F'_i(x)\}$. Since F_i and F'_i are nondecreasing, then it must be that y_i is in the support of F_i . From Proposition 11, let $a \leq \min\{\bar{x}_1, \bar{x}_2, \varpi\}$ be such that each F_i is continuous and strictly increasing on $[\underline{x}, a)$ and constant on $[a, \min\{\bar{x}_1, \bar{x}_2, \varpi\})$. We will consider two cases corresponding to whether $y_i < a$ for some firm i or $y_i \geq a$ for each firm i .

Case 1: $y_i < a$ for some firm i

In this case, Proposition 11 implies that y_i is in the support of F_j . Then $\pi_j = \lim_{x \rightarrow y^-} u_j(x, F_i)$. By definition of equilibrium, $\pi'_j \geq \lim_{x \rightarrow y^-} u'_j(x, F'_i)$. Note that for all $x < y_i$ such that $\mu_j(\{x\}) = \mu'_j(\{x\}) = 0$,

$$\begin{aligned} u'_j(x, F'_i) &= (1 - F'_i(x))\varphi'_j(x) + \int_{[\underline{x}', x]} \psi'_j(x, p_i) dF'_i \\ &\geq (1 - F'_i(x))\varphi_j(x) + F'_i(x) \int_{[\underline{x}', x]} \psi_j(x, p_i) d\nu'_i(x), \end{aligned}$$

where $\nu'_i(x)$ is the conditional distribution of μ'_i given that $p_i < x$. Lemma 7 implies that $\int_{[\underline{x}', x]} \psi_j(x, p_i) d\nu'_i(x) \geq \int_{[\underline{x}', x]} \psi_j(x, p_i) d\nu_i(x)$. Thus

$$\begin{aligned} u'_j(x, F'_i) &\geq (1 - F'_i(x))\varphi_j(x) + F'_i(x) \int_{[\underline{x}', x]} \psi_j(x, p_i) d\nu_i(x) \\ &= (1 - F_i(x))\varphi_j(x) + F_i(x) \int_{[\underline{x}', x]} \psi_j(x, p_i) d\nu_i(x) \\ &\quad - (F'_i(x) - F_i(x)) \left(\varphi_j(x) - \int_{[\underline{x}', x]} \psi_j(x, p_i) d\nu_i(x) \right) \\ &= \pi_j - (F'_i(x) - F_i(x)) \left(\varphi_j(x) - \int_{[\underline{x}', x]} \psi_j(x, p_i) d\nu_i(x) \right). \end{aligned}$$

Note that

$$\begin{aligned} &\lim_{x \rightarrow y_i^-} (F'_i(x) - F_i(x)) \left(\varphi_j(x) - \int_{[\underline{x}', x]} \psi_j(x, p_i) d\nu_i(x) \right) \\ &= \lim_{x \rightarrow y_i^-} (F'_i(x) - F_i(x)) \left(\varphi_j(y_i) - \int_{[\underline{x}', y_i]} \psi_j(y_i, p_i) d\nu_i(x) \right). \end{aligned}$$

Since $\varphi_j(y_i) - \int_{[\underline{x}', y_i]} \psi_j(y_i, p_i) d\nu_i(x) > 0$, then if $\lim_{x \rightarrow y_i^-} (F'_i(x) - F_i(x)) = 0$, then $\pi'_j \geq \pi_j$. This would be a contradiction, and so it must be that $\lim_{x \rightarrow y_i^-} (F'_i(x) - F_i(x)) > 0$. Thus, it must be that $\mu_i(\{y_i\}) > 0$, contradicting Proposition 11.

Case 2: $y_i \geq a$ for each firm i

In this case, Proposition 11 implies that $y_i \geq \varpi$ for each firm i . Without loss of generality, assume that $\bar{x}_1 \leq \bar{x}_2$ and that $\mu_1(\{\bar{x}_2\}) = 0$. Since \bar{x}_2 is in the support of F_2 , then

$$\begin{aligned} \pi_2 &= \int_{[\underline{x}, \bar{x}_2]} \psi_2(\bar{x}_2, p_1) dF_1 \\ &= \int_{[\underline{x}, y_1]} \psi_2(\bar{x}_2, p_1) dF_1 \\ &= F_1(y_1) \int_{[\underline{x}, y_1]} \psi_2(\bar{x}_2, p_1) d\nu_1, \end{aligned}$$

where ν_1 is the conditional distribution of F_1 given that $p_1 \leq y_1$. From Lemma 7, since $F_1'(x) > F_1(x)$ for all $x < y_1$, then

$$\int_{[\underline{x}, y_1)} \psi_2(\bar{x}_2, p_1) d\nu_1 \leq \int_{[\underline{x}, y_1)} \psi_2(\bar{x}_2, p_1) d\nu_1'.$$

Note that by definition of equilibrium,

$$\begin{aligned} \pi_2' &\geq \lim_{x \rightarrow \bar{x}_2^-} u_2(x, F_1') \\ &= (1 - F_1'(\bar{x}_2))\varphi(\bar{x}_2) + F_1'(y_1) \int_{[\underline{x}, y_1)} \psi_2(\bar{x}_2, p_1) d\nu_1' \\ &\geq (1 - F_1'(\bar{x}_2))\varphi(\bar{x}_2) + F_1'(y_1) \int_{[\underline{x}, y_1)} \psi_2(\bar{x}_2, p_1) d\nu_1. \end{aligned}$$

From the construction of y_1 , y_1 is in the support of F_1 and $F_1(y_1) \geq F_1'(y_1)$. Furthermore, from the previous part of the proof, it must be that F_1 is atomless on $[\underline{x}, \bar{x}_1]$. It follows that $F_1(y_1) = F_1(\{y_1\})$. Therefore,

$$\begin{aligned} \pi_2' &\geq (1 - F_1'(\bar{x}_2))\varphi(\bar{x}_2) + F_1(y_1) \int_{[\underline{x}, y_1)} \psi_2(\bar{x}_2, p_1) d\nu_1 \\ &\geq \pi_2. \end{aligned}$$

This contradicts the fact that $\pi_2' < \pi_2$. We conclude that all equilibria are payoff equivalent.

■

6.2 Independent Residual Profits

Definition 1 *A BE game has independent residual profit if for both firms i , $\psi_i(p_i, p_j) = \psi_i(p_i, p_j')$ for all $p_j, p_j' \leq p_i$.*

The commonly studied BE game with constant marginal cost of production and efficient demand rationing is a prominent example of a BE game with independent residual profit. Despite the inherent similarities between efficient demand rationing and independent residual profits, the two concepts are not equivalent. In fact, neither concept implies the other. A game with efficient rationing and strictly convex cost up to capacity does not have independent residual profit. Moreover, it is possible to construct a game with convex costs in which the rationing rule is not efficient and the λ_i 's are chosen to be decreasing in such a way that the residual quantities are constant, thereby satisfying independent residual profit.

The following proposition is a characterization of the unique equilibrium of a BE game with independent residual profit. In this case, we will abuse notation and write the residual profit as $\psi_i(p_i)$.

Proposition 13 *Suppose that the game has independent residual profit, each firm has a unique maximizer \tilde{p}_i , that each $\psi_i(p_i, p_j)$ is weakly increasing in p_i on $[p_j, \tilde{p}_i]$, and that $\tilde{p}_i \leq \tilde{p}_j$ whenever $\bar{r}_i < \bar{r}_j$. Then there is a unique equilibrium. The equilibrium profit for each firm i is $u_i^* = \bar{\varphi}_i$ and the equilibrium strategy for each firm i is the (possibly degenerate) cumulative distribution function defined by*

$$F_i(x) = \frac{\varphi_j(x) - \bar{\varphi}_j}{\varphi_j(x) - \psi_j(x)},$$

on $[\bar{r}, \min\{\tilde{p}_1, \tilde{p}_2\})$ and $F(\min\{\tilde{p}_1, \tilde{p}_2\}) = 1$, where j is the firm other than i .

Proof of Proposition 13. The proof that $u_i^* = \bar{\varphi}_i$ is an obvious corollary to Proposition 8 since, in this case $\underline{r} = \bar{r}$. It remains to be shown that the equilibrium is unique. Based on the proof of Proposition 2, we know that if the equilibrium is in pure strategies, then it must be unique. It only remains to rule out the case of multiple mixed strategy equilibria. The remainder of the proof is constructive and follows a similar argument to the proof of Theorem 3 in Siegel (2010).

We begin by showing that each firm's equilibrium strategy must have full support on $(\underline{x}, \min\{\tilde{p}_1, \tilde{p}_2\})$. Suppose to the contrary that there is some firm i and interval $(a, b) \subset (\underline{x}, \min\{\tilde{p}_1, \tilde{p}_2\})$ with $F_i(a) = F_i(x)$ for all $x \in (a, b)$ and $F_i(x) < F_i(a)$ for all $x < a$. We will consider two cases corresponding to whether $F_i(a) < 1$ or $F_i(a) = 1$.

Case 1: $F_i(a) < 1$

Note that $\int u_j(x, p_i) d\mu_i = \varphi_j(x)(1 - F_i(a)) + \psi_j(x)F_i(a)$ for all $x \in (a, b)$. Since $b \leq \min\{\tilde{p}_1, \tilde{p}_2\}$, then ψ_j is weakly increasing in p_j on (a, b) . Since $F_i(a) < 1$, then it follows that $\int u_j(x, p_i) d\mu_i$ is strictly increasing in x on (a, b) . Thus, it must be that $F_j(a) = F_j(x)$ for all $x \in (a, b)$. If $F_j(x) = F_j(a)$ for some $x < a$, then we could reiterate this logic to show that $F_i(x) = F_i(a)$, a contradiction. Thus, $F_j(x) < F_j(a)$ for all $x < a$.

If $a = \underline{\rho}$ then $\mu_i(\{a\}) > 0$. Note that $\int u_i(\tilde{p}_i, p_j) d\mu_j \geq \psi_i(\tilde{p}_i) > \psi_i(a) = \int u_i(a, p_j) d\mu_j$, contradicting a as a best response. Thus, $a > \underline{\rho}$. Suppose that $\mu_i(\{a\}) = 0$, implying that F_i is continuous at a . Note that at any price x , $\int u_j(x, p_i) d\mu_i \leq \varphi_j(x)(1 - F_i(x)) + \psi_j(x)F_i(x)$, which is continuous at $x = a$. Choose any $y \in (a, b)$. As noted above, $\int u_j(x, p_i) d\mu_i$ is strictly increasing on (a, b) , and since $\mu_i(\{a\}) = 0$, it follows that $\int u_j(y, p_i) d\mu_i > \int u_j(a, p_i) d\mu_i$. Let $\varepsilon > 0$ be such that $\varepsilon < \int (u_j(y, p_i) - u_j(a, p_i)) d\mu_i$ and choose $\delta > 0$ such that $|\int (u_j(x, p_i) - u_j(a, p_i)) d\mu_i| < \varepsilon$ for all $x \in (a - \delta, a)$. Then note that for all $x \in (a - \delta, a]$,

$$\begin{aligned} \int (u_j(y, p_i) - u_j(x, p_i)) d\mu_i &\geq \int (u_j(y, p_i) - u_j(a, p_i) + u_j(a, p_j) - u_j(x, p_i)) d\mu_i \\ &> \int (u_j(y, p_i) - u_j(a, p_i) - \varepsilon) d\mu_i \\ &> 0. \end{aligned}$$

Thus, $(a - \delta, a]$ contains no best responses for firm j , and so $F_j(x) = F_j(a)$ for all $x \in (a - \delta, a]$, a contradiction. It therefore must be the case that $\mu_i(\{a\}) > 0$. From Lemma 1, we know that $\mu_j(\{a\}) = 0$. If $F_j(a) < 1$, then swapping the roles of i and j and applying the previous analysis shows a contradiction. Thus, it must be that $F_j(a) = 1$. It follows that $\int u_i(a, p_j) d\mu_j = \psi_i(a) < \psi_i(\tilde{p}_i) = \int u_i(\tilde{p}_i, p_j) \mu_j$, contradicting a as a best response.

Case 2: $F_i(a) = 1$

Suppose that $a = \underline{\rho}$. Then $u_j(\underline{\rho}, F_i) = \psi_j(\underline{\rho}) < \psi_j(\tilde{p}_j) = u_j(\tilde{p}_j, F_i)$. It follows that $\underline{x}_j > \underline{x}_i$, contradicting Lemma 2. Suppose instead that $a > \underline{\rho}$. Then from Lemma 1, at most one firm's strategy may have an atom at a . If $\mu_j(\{a\}) > 0$, then $\mu_i(\{a\}) = 0$, so $\int u_j(a, p_i) d\mu_i = \psi_j(a) < \psi_j(\tilde{p}_j) = \int u_j(\tilde{p}_j, p_i) d\mu_i$, violating F_i as an equilibrium strategy. Thus, $\mu_j(\{a\}) = 0$. Note that $\int u_j(x, p_i) d\mu_i = \psi_j(x)$ for all $x > a$, and so it must be that $\int u_j(\tilde{p}_j, p_i) d\mu_i > \int u_j(x, p_i) d\mu_i$ for all $x \in (a, \tilde{p}_j)$. Thus, $F_j(x) = F_j(a)$ for all $x \in (a, \tilde{p}_j)$ and $F_j(a) < 1$, and so the previous case applies. We conclude that the equilibrium strategies have full support on $(\underline{x}, \min\{\tilde{p}_1, \tilde{p}_2\})$.

We next show that each firm's strategy is continuous on $(\underline{x}, \min\{\tilde{p}_1, \tilde{p}_2\})$. The proof of this statement is identical to the proof that neither firm may have an atom at \underline{x} , as shown by Lemma 2. Suppose to the contrary that some firm j has an atom $\mu_j(\{a\}) > 0$ at some price $a \in (\underline{x}, \min\{\tilde{p}_1, \tilde{p}_2\})$. Since each firm's strategy has full support, it must be that $\mu_i((a, a + \delta)) > 0$ for all $\delta > 0$. It is sufficient to show that there is some $x' < a$ and neighborhood $(a, a + \delta)$ such that $\int u_i(x', p_j) d\mu_j > \int u_i(x, p_j) d\mu_j$ for all $x \in (a, a + \delta)$.

Let $\varepsilon > 0$ be such that $\varepsilon < (1/3)\mu_j(\{a\})(\varphi_i(a) - \psi_i(a))$. Since φ_i is continuous on $[0, \tilde{p}_i]$, it must also be uniformly continuous. Let $\delta > 0$ be such that (i) if $|x - x'| < \delta$, then $|\varphi_i(x) - \varphi_i(x')| < \varepsilon$ and (ii) if $|x - a| < \delta$, then $|\psi_i(x) - \psi_i(a)| < \varepsilon$. Then consider the price $x' = a - (1/4)\delta$ and neighborhood $(a, a + (1/2)\delta)$. Then note that for all $x \in (a, a + \delta)$,

$$\begin{aligned} u_i(x', p_j) - u_i(x, p_j) &= \begin{cases} \varphi_i(x') - \psi_i(x, p_j) & \text{if } p_j < x \\ \varphi_i(x') - [\alpha_i(x, x)\varphi_i(x) + (1 - \alpha_i(x, x))\psi_i(x, p_j)] & \text{if } p_j = x \\ \varphi_i(x') - \varphi_i(x) & \text{if } p_j > x \end{cases} \\ &\geq \begin{cases} \varphi_i(x') - \psi_j(x, x) & \text{if } p_j < x \\ \varphi_i(x') - \varphi_i(x) & \text{if } p_j \geq x \end{cases} \\ &= \begin{cases} \varphi_i(x') - \varphi_i(a) + \varphi_i(a) - \psi_i(a) + \psi_i(a) - \psi_i(x) & \text{if } p_j < x \\ \varphi_i(x') - \varphi_i(x) & \text{if } p_j \geq x \end{cases} \end{aligned}$$

Since $\max\{|x' - a|, |x' - x|, |x - a|\} < \delta$, it follows that $\min\{\varphi_i(x') - \varphi_i(a), \varphi_i(x') - \varphi_i(x), \psi_i(a) - \psi_i(x)\} > -\varepsilon$. Thus, for all $x \in (a, a + \delta)$,

$$u_i(x', p_j) - u_i(x, p_j) > \begin{cases} \varphi_i(a) - \psi_i(a) - 2\varepsilon & \text{if } p_j < x \\ -\varepsilon & \text{if } p_j \geq x \end{cases}.$$

It follows that for all such x ,

$$\int (u_i(x', p_j) - u_i(x, p_j)) d\mu_j > \int_{[\underline{x}, x]} (\varphi_i(a) - \psi_i(a) - 2\varepsilon) d\mu_j - \int_{[x, \bar{x}]} \varepsilon d\mu_j.$$

As ε was chosen so that such that $2\varepsilon < \varphi_i(\underline{x}) - \psi_i(a)$, it follows that

$$\begin{aligned} \int (u_i(x', p_j) - u_i(x, p_j)) d\mu_j &> \mu_j(\{a\}) (\varphi_i(a) - \psi_i(a) - 2\varepsilon) - \varepsilon \\ &> \mu_j(\{a\}) (\varphi_i(a) - \psi_i(a)) - 3\varepsilon. \end{aligned}$$

Thus, given our assumption on ε , $\int (u_i(x', p_j) - u_i(x, p_j)) d\mu_j > 0$, and so x' is a profitable deviation from all $x \in (a, a + \delta)$. Thus, $(a, a + \delta)$ contains no best responses for firm i , contradicting the equilibrium strategies as having full support. We conclude that each F_i is continuous on $(\underline{x}, \min\{\tilde{p}_1, \tilde{p}_2\})$.

Since the strategies are continuous, the expected profit of firm i at any price $x \in (\underline{x}, \min\{\tilde{p}_1, \tilde{p}_2\})$ is

$$\int u_i(x, p_j) d\mu_j = \varphi_i(x)(1 - F_j(x)) + \psi_i(x)F_j(x).$$

Furthermore, since each $u_i^* = \bar{\varphi}_i$, then it must be that for all $x \in (\underline{x}, \min\{\tilde{p}_1, \tilde{p}_2\})$, $\varphi_i(x)(1 - F_j(x)) + \psi_i(x)F_j(x) = \bar{\varphi}_i$. This equation can easily be solved to find that

$$(F_1(x), F_2(x)) = \left(\frac{\varphi_2(x) - \bar{\varphi}_i}{\varphi_2(x) - \psi_2(x)}, \frac{\varphi_1(x) - \bar{\varphi}_i}{\varphi_1(x) - \psi_1(x)} \right),$$

and by the right continuity of CDF's, these must be the equilibrium strategies on $[\underline{x}, \min\{\tilde{p}_1, \tilde{p}_2\})$.

It remains to be shown that no player j will ever choose a price $p_j > \min\{\tilde{p}_1, \tilde{p}_2\}$. Without loss of generality, suppose that $\tilde{p}_j = \min\{\tilde{p}_1, \tilde{p}_2\}$. If $\bar{r}_i = \bar{r}$, then $\bar{\varphi}_j = \psi_j(\tilde{p}_j)$, and so $\psi_i(p_j) < \psi_j(\min\{\tilde{p}_1, \tilde{p}_2\})$ for all $p_j > \min\{\tilde{p}_1, \tilde{p}_2\}$. In this case, $\lim_{x \rightarrow \tilde{p}_j^-} \mathcal{M}_i(x) = 1$, so firm i does not play prices higher than $\min\{\tilde{p}_1, \tilde{p}_2\}$. It follows that firm j receives its residual profit with certainty at any price $p_j \geq \tilde{p}_j$, and thus would never price higher than $\min\{\tilde{p}_1, \tilde{p}_2\}$. Otherwise, $\bar{r}_i < \bar{r}_j$, so $\bar{\varphi}_i = \psi_i(\tilde{p}_i)$. Since $\tilde{p}_i \leq \tilde{p}_j$, then $\tilde{p}_i = \tilde{p}_j$. This further implies that $\lim_{x \rightarrow \tilde{p}_j^-} \mathcal{M}_j(x) = 1$, and so firm j does not play prices higher than $\min\{\tilde{p}_1, \tilde{p}_2\}$. It follows that firm i receives its residual profit with certainty when choosing any price $p_i \geq \tilde{p}_i$, and so will never choose a price higher than $\min\{\tilde{p}_1, \tilde{p}_2\}$. ■

It is worth highlighting the significance of the fact that the critical safe price is equal to the critical judo price. This implies that at least one of the two firms earns at most its max-min payoff in equilibrium (both if firms have the same safe prices), implying that rents are maximally dissipated in equilibrium. This corresponds exactly with the equilibria of the contests studied by Siegel (2009). This correspondence is to be expected, as the

most important difference between our model and the traditional all-pay contest is that the players' payoffs conditional on losing the contest is a function of the winner's strategy, and this is no longer the case under the assumption of independent residual profit. The remaining distinction is that firms in our model may be able to obtain a positive payoff even if they are unable to guarantee that they will have the lowest price.

7 Discussion

The methodology we have used to provide this characterization involves a realization of the payoffs abstractly as front-end and residual profits. This abstraction allows for simplified analysis, and more importantly, demonstrates the connection between the literature on price competition and all-pay contests. These classes of games exhibit similar characteristics, and the methodology used here should be applicable to more general game structures that encompass both of these classes. In the following discussion we briefly sketch the extension of our analysis to models with demand uncertainty or more than two firms. In doing this we detail both the results that should easily generalize and those that do not.

In terms of the addition of demand uncertainty, the main change is in the interpretation of the front-side and residual profit functions φ_i and ψ_i . If we suppose that these are expected profit functions over some set of uncertain demand states, then much of the primary results generalizes to this setting. A substantive difference is that the point of indifference between the front-side and residual profits, $\underline{\rho}$, becomes a function of the demand state. As such, the only pure strategy equilibrium candidate is at the lowest possible value for $\underline{\rho}$, as otherwise firms would have incentive to undercut one another. However, pricing at such a level would never be optimal, as in expectation the firms could greatly benefit from raising their prices. Thus, the addition of demand uncertainty can only further contribute to the fragility of pure strategy equilibrium in pricing games. One result that is lost in the generalization is the classification of pure strategy equilibria, however, given its fragility, the inability to obtain such a result seems inconsequential. Regarding the characterization of mixed strategy equilibrium, most of the results on payoffs and pricing bounds should extend straightforwardly to a setting with demand uncertainty.¹⁸ The uniqueness results of Section 6 also generalize to the demand uncertainty setting as long as the expected profit functions satisfy Assumptions 5 and 6. If each demand state's front-side profit is concave, then Assumption 6 holds for the expected profit. Interpreting the restrictions on the profit of demand state is more difficult for Assumption 5 which restricts expected residual profit to be strictly concave.

Understanding how the results can be extended to the case of oligopoly is far more difficult. The n firm oligopoly pricing game can be viewed as an $n - 1$ heterogenous prize

¹⁸Note, this requires the expected profit functions to satisfy analogous properties to those outlined in Assumptions 1-4, which inherently puts more restrictions on the profit function for each demand state.

contest in which each player's prize payoffs dependent on others bids and non-monotonic in its own bid.¹⁹ The primary result on pure strategy equilibrium (Proposition 1) generalizes trivially to the oligopoly setting. We do not include this more general result in the present work as its exposition would require a great deal of additional notation that would only serve to make the current analysis less salient. The extension of the remainder of the results is not clear. We believe that a general equilibrium expected payoff characterization similar to Proposition 8 should hold for oligopoly, however, this requires notions of judo and safe prices, and there may be multiple critical levels of these prices that depend on which firms are competing for the lowest price.

8 Appendix

8.1 Existence of Equilibrium (Proof of Proposition 7)

The following definition is from Allison and Lepore (2014). Let X_i and u_i denote player i 's strategy set and utility function, respectively. Define the discontinuity mapping $D_i : X_i \rightarrow X_{-i}$ such that

$$D_i(x_i) = \{x_{-i} \in X_{-i} : u_i(x_i, x_{-i}) \text{ is discontinuous in } x_{-i} \text{ at } (x_i, x_{-i})\}.$$

Definition 2 *A game satisfies disjoint payoff matching (DPM) if for each player i and all $x_i \in X_i$, there exists a sequence $\{x_i^k\} \subset X_i$ such that:*

- 1) $\liminf_k u_i(x_i^k, x_{-i}) \geq u_i(x_i, x_{-i})$ for all $x_{-i} \in X_{-i}$, and
- 2) $\limsup_k D_i(x_i^k) = \emptyset$.²⁰

Fact 1 (Allison and Lepore (2014)) *If each supply function s_i is continuous, then the game possesses a (possibly mixed) Nash equilibrium.*

The problem in using this definition of DPM to verify existence of equilibrium when the functions s_i are discontinuous is that it may be impossible to satisfy part 2 of the definition, as the discontinuities in a firm's supply function induce discontinuities in the other firm's payoff. A trivial modification is sufficient to generalize the existence result. Define the discontinuity map $D'_i : X_i \rightarrow X_{-i}$ such that

$$D'_i(x_i) = \{x_{-i} \in D_i(x_i) : u_i(x_i, x_{-i}) \text{ is not lower semicontinuous in } x_{-i} \text{ at } (x_i, x_{-i})\}.$$

¹⁹Xiao (2016) considers a heterogeneous prize all-pay contest in which the payoff associated with each prize is decreasing in a player's bid (monotonic) and no other player's bid impacts a player's prize value.

²⁰Here the limit superior of sequence of sets A_k refers to the set $\bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$.

By replacing D_i with D'_i in the definition of DPM, the proof of the main result of Allison and Lepore (2014) remains valid. Moreover, since the supply functions are upper semicontinuous, then the residual profit ψ_i is lower semicontinuous in p_j . It follows that the discontinuity sets $D_i(x_i)$ and $D'_i(x_i)$ coincide, and so our game satisfies this modified definition of DPM.

8.2 Technical Lemmas: Uniqueness of Equilibrium with Identical Firms

Proof of Lemma 5. Let F_i be constant on some interval $[a, b) \subset [\underline{x}, \min\{\bar{x}_1, \bar{x}_2, \varpi_1, \varpi_2\})$. Then for all $x \in (a, b)$,

$$u_j(x, F_i) = (1 - F_i(a))\varphi_j(x) + \int_{[0, a]} \psi_j(x, p_i) dF_i.$$

Since $a < \varpi_j$, then $\psi_j(a, a) > 0$. Since ψ_j is nonincreasing in p_i and $b < \varpi_j$, it follows that $\psi_j(x, p_i) > 0$ for all $x \in (a, b)$ and all $p_i \leq a$. Thus, by assumption,

$$\int_{[0, a]} \psi_j(x, p_i) dF_i$$

is concave in x on (a, b) . Moreover, since $a < \bar{x}_i$, then $F_i(a) < 1$, so we may conclude that $u_j(x, F_i)$ is strictly concave on (a, b) . Therefore, either $u_j(a, F_i) > u_j(x, F_i)$ for all $x \in (a, b)$ or there is some $\tilde{x} \in (a, b)$ such that $u_j(\tilde{x}, F_i) > u_j(x, F_i)$ for all $x \in (a, \tilde{x})$. In either case, it must be that F_j is constant on some interval $[a, b')$. ■

Proof of Lemma 6. To avoid confusion, we will use $-i$ in this proof to refer to the firm other than i , allowing j to represent an arbitrary firm. Let F_1 and F_2 be constant on some interval $[a, b)$ where $b < \min\{\bar{x}_1, \bar{x}_2, \varpi_1, \varpi_2\}$, a is in the support of F_1 , and b is in the support of F_j for some j . As argued in the proof of Lemma 5, it must be that each $u_i(x, F_{-i})$ is strictly concave on (a, b) . Define $\tilde{u}_i(x, F_{-i})$ to be the continuous extension of $u_i(x, F_{-i})$ from (a, b) to $[a, b]$. Since $[a, b]$ is compact, then $\tilde{u}_i(x, F_{-i})$ has a unique maximizer \tilde{x}_i on $[a, b]$. We will show that $\tilde{x}_i = b$ and $\tilde{x}_{-i} = a$ for some firm i .

Note first that since $\lim_{x \rightarrow b^-} u_i(x, F_{-i}) \geq u_i(b, F_{-i}) \geq \lim_{x \rightarrow b^+} u_i(x, F_{-i})$, then if $\tilde{x}_i < b$ for each firm i , then $\lim_{x \rightarrow b^-} u_i(x, F_{-i}) < u_i(\tilde{x}_i, F_{-i})$ for both firms. Thus, b could not be a best response for either firm. It follows that $\tilde{x}_i = b$ for some firm i . This implies that $u_i(x, F_{-i})$ is strictly increasing on (a, b) . We will argue that $\mu_i(\{a\}) = 0$. Suppose to the contrary that $\mu_i(\{a\}) > 0$. If $a > \underline{\rho}$, then from Lemma 1, it must be that $\mu_{-i}(\{a\}) = 0$, and so $u_i(a, F_{-i}) = \tilde{u}_i(a, F_{-i})$. The same is true when $a = \underline{\rho}$ since u_i is continuous at $\underline{\rho}$. In either case, it follows that $u_i(a, F_{-i}) < \lim_{x \rightarrow \tilde{x}_i^-} u_i(x, F_{-i})$, violating a as a best response. We conclude that $\mu_i(\{a\}) = 0$.

Next, since $\mu_i(\{a\}) = 0$, then $\lim_{x \rightarrow a^-} u_{-i}(x, F_i) = u_{-i}(a, F_i) = \lim_{x \rightarrow a^+} u_{-i}(x, F_i)$. If $\tilde{x}_{-i} > a$, then $u_{-i}(\tilde{x}_{-i}, F_i) > u_{-i}(x, F_i)$ for all $x \in (a - \delta, \tilde{x}_{-i})$ for some $\delta > 0$, in which case

a would not be in the support of F_{-i} . This would further imply that a is not in the support of F_i since $\tilde{x}_j = b$, but this contradicts the assumption that a is in the support of F_1 . We conclude that $\tilde{x}_{-i} = a$.

It remains to be shown that $\mu_i(\{b\}) > 0$. Suppose to the contrary that $\mu_i(\{b\}) = 0$. The fact that $u_{-i}(x, F_i)$ is strictly decreasing on (a, b) implies that $\mu_{-i}(\{b\}) = 0$, and so $u_i(x, F_{-i})$ is continuous in x at $x = b$. Thus, there is a neighborhood $(b - \delta, b + \delta)$ such that $|u_i(x, F_j) - u_i(b, F_j)| < u_i(a, F_j) - u_i(b, F_j)$ for all $x \in (b - \delta, b + \delta)$. It follows that $u_i(a, F_j) > u_i(x, F_j)$ for all $x \in (b - \delta, b + \delta)$, and so F_i is constant on $[a, b + \delta)$. For sufficiently small δ , $b + \delta < \min\{\varpi_j, \bar{x}_1\}$ and we may conclude that $u_j(x, F_i)$ is strictly concave on $[a, b + \delta)$. There can be at most one maximizer of $u_j(x, F_i)$ on $(a, b + \delta)$, and so the fact that b is in the support of F_j implies that $\mu_j(\{b\}) > 0$. ■

Proof of Lemma 7. Suppose that $F(x) \leq G(x)$ for all $x \in [a, b]$. Since f is monotonic, then f has at most countably many discontinuities. Let $\{y_n\} \subset [a, b]$ be a sequence that includes all discontinuities of f .

For each $n \in \mathbb{N}$, Let $\Pi_n = \{x_0^n, \dots, x_{m(n)}^n\}$ be a finite partition of X with $\Pi_n \subset \Pi_{n+1}$, $|\Pi_n| < 1/n$, $y_1, \dots, y_n \in \Pi_n$, and $x_0^n = a$, $x_{m(n)}^n = b$. Further, define

$$\alpha_n(x) = f(x_i^n) \text{ where } x \in [x_{i-1}^n, x_i^n).$$

We will show that $\alpha_n \rightarrow f$ on $[a, b]$. Let $x \in [a, b]$. If f is discontinuous at x , then $x = y_m$ for some m . By construction, $\alpha_n(x) = f(x)$ for all $n \geq m$, so clearly $\alpha_n(x) \rightarrow f(x)$. Alternatively, suppose that f is continuous at x , then define $z^n(x) = x_i^n$, where $x \in [x_{i-1}^n, x_i^n)$ and $x_{i-1}^n, x_i^n \in \Pi_n$. Thus, $\alpha_n(x) = f(z^n(x))$. Then note that $z^n(x) \rightarrow x$, and since f is continuous at x , it follows immediately that $\alpha_n(x) \rightarrow f(x)$.

Since f is positive and nonincreasing, $|f| \leq |f(a)|$. Thus, the Lebesgue dominated convergence implies that for any measure μ , $\lim_n \int \alpha_n d\mu = \int \lim_n \alpha_n d\mu = \int f d\mu$. We conclude that $\int \alpha_n dF, \int \alpha_n dG \rightarrow \int f dF$. It is therefore sufficient to show that $\int \alpha_n dF \leq \int \alpha_n dG$ for all n .

Let μ be the measure associated with F and λ the measure associated with G . Note that

$$\int \alpha_n dF = \sum_{i=1}^{m(n)} f(x_i^n) \mu([x_{i-1}^n, x_i^n)).$$

For notational convenience, let $\mu_i^n = \mu([x_{i-1}^n, x_i^n])$ and $f_i^n = f(x_i^n)$. Then note that

$$\begin{aligned} \sum_{i=1}^{m(n)} f_i^n \mu_i^n &= \mu_1^n (f_1^n - f_2^n) \\ &\quad + (\mu_1^n + \mu_2^n) (f_2^n - f_3^n) \\ &\quad \vdots \\ &\quad + (\mu_1^n + \mu_2^n + \dots + \mu_{m(n)-1}^n) (f_{m(n)-1}^n - f_{m(n)}^n) \\ &\quad + f_{m(n)}^n \\ &= f_{m(n)}^n - \sum_{i=1}^{m(n)-1} (f_{i+1}^n - f_i^n) F(x_i^n). \end{aligned}$$

Thus,

$$\int \alpha_n dF - \int \alpha_n dG = \sum_{i=1}^{m(n)-1} (f_{i+1}^n - f_i^n) (G(x_i^n) - F(x_i^n)).$$

Since $\Pi_n \subset \Pi_{n+1}$, then it must be that $f_{i+1} \geq f_i$. Further, since $F \leq G$, the equation above implies that $\int \alpha_n d\mu - \int \alpha_n d\lambda \leq 0$, or rather, $\int \alpha_n d\mu \leq \int \alpha_n d\lambda$. ■

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